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The indecomposable K_3 of rings and homology of SL_2

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Abstract

We study the relationship between the third homology group of $SL_2(R)$ and $K_3(R)$, for R in a large class of rings. These results extend a previous work of C.-H. Sah.

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0. Introduction

The present work investigates the relationship between the K-theory and the homology of linear groups in low degrees. The motivations and applications of this problem are explained in [10, 11]. The principal results are summarized below. Let R be a commutative H1-ring (e.g. a semi-local ring with infinite residue fields). We have,

Theorem 1.22. *The morphism* $H_3(GL_2(R);\mathbb{Q}) \to H_3(GL_3(R);\mathbb{Q})$ *induced by* $GL_2(R) \to GL_3(R)$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, *is injective.*

In connection with algebraic K-theory, if K_*^M denotes the Milnor K-theory of R, define the indecomposable part $K_3(R)^{\text{ind}}$ to be $K_3(R)/\operatorname{Im} K_3^M$. We have,

Theorem 2.2. *Let R be a (commutative)* H1*-ring. We have an isomorphism*:

$$K_3(R)^{\text{ind}}_{\mathbb{Q}} \cong H_0(R^\times; H_3(SL_2(R);\mathbb{Q})).$$

Moreover, by (2.5), we can identify the indecomposable K_3 of R with the weight two part of the "Adams decomposition" of K_3.

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The present work is organized as follows: Section 1 is devoted to the proof of the injectivity of the morphism

$$\mathrm{H}_3(GL_2(R);\mathbb{Q}) \to \mathrm{H}_3(GL_3(R);\mathbb{Q}),$$

induced by the map $GL_2(R) \to GL_3(R)$, $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The complete argument is fairly complicated and the "dévissage" of the spectral sequence involved is carefully explained. Section 2 is devoted to establishing the relationship between the previous results and K-theory. Section 3 is a short comment on related works and problems. In the appendix all the specific tools that we use in this paper can be found.

1. The injectivity of $\mathrm{H}_3(GL_2(R);\mathbb{Q}) \to \mathrm{H}_3(GL_3(R);\mathbb{Q})$

Recall that if R is a ring and M a right R-module, then an element u of M is said to be unimodular if there exists a linear form $f : M \to R$ such that $f(u) = 1$. We will let $Um(M)$ denote the set of unimodular elements in M. The stable rank of R, denoted by $sr(R)$, is the smallest integer $n \geq 1$ such that the following property holds:

$$(\mathrm{SR}_n)\begin{cases} \text{for all } (a_1,\dots,a_{n+1}) \in Um(R^{n+1}), \text{ there exist } b_1,\dots,b_n, \\ \text{in } R, \text{ such that } (a_1 + a_{n+1}b_1,\dots,a_n + a_{n+1}b_n) \in Um(R^n). \end{cases}$$

If such integer does not exist, we set $sr(R) = \infty$. A ring R is said to be an H1-ring (see [13]) if the following property holds for all integers k, l such that $l \geq 2$, $k \geq 1$: for any family of k surjective linear forms $f_i : R^l \to R$, there exists $v \in R^l$ such that $f_i(v) \in R^\times$, for $i = 1,\dots,k$.

A ring R is called an $S(n)$-ring (see [17]), $n \geq 2$, if there exists a family of n invertible elements of the center of R, such that all partial sums formed with these elements are invertibles. We say that the ring is $S(\infty)$ if it is an $S(n)$-ring for all n, $n \geq 2$.

Remark 1.1. (1) The notion of H1-ring was introduced by Guin in [13] to get a stability result for the general linear group with trivial coefficients, and used by Akef [1] to get a stability result with coefficient in the adjoint action.

(2) The notion of $S(n)$-ring appears in the work of Nesterenko and Suslin [17] and allows them to relate the homology of GL_n of an $S(\infty)$-ring with the homology of its affine group and deduce from this a result of homological stability (depending of the stable rank) for the general linear group over this kind of ring.

(3) If R is H1, then $sr(R) = 1$.

Example 1.2. The fundamental example of H1-ring is a semi-local ring with infinite residue fields. The division algebra of real quaternion is H1. Notice that in the commutative case the condition H1 implies $S(\infty)$.

Before going further, we want to introduce a more geometric way for understanding H1-rings. Let R be an H1-ring. We call a free direct summand of R^n, $n \in \mathbb{N} - \{0\}$, a *subspace* of R^n. A subspace of rank $n-1$ will be called an *hyperplane* of R^n, and a rank one subspace will be called a *line*. So the geometric interpretation of an H1-ring is the following: *let $n \geq 2$ and let H_i, with $i = 1,\dots,m$, be a (finite) family of hyperplanes in R^n. Then there exists a line L which is a common complement to each H_i.* For fields, if R is H1, we denote by $\mathbb{P}^n(R)$ the set of lines in R^{n+1}, and we call it n-dimensionnal projective space. In the following we assume that R is an H1-ring.

Definition 1.3. A $(k+1)$-tuple $(v_0,\dots,v_k)$ of points in $\mathbb{P}^n(R)$ is said m-generic, for $m \leq n$, if for all $t \leq \min(m,k)$, every subset of $\{v_0,\dots,v_k\}$ with $(t+1)$ elements spans a t-dimensionnal projective subspace in $\mathbb{P}^n(R)$ (i.e. a subspace of dimension $t+1$ in R^{n+1}).

Note that for H1-rings, we have the famous result.

Proposition 1.4. *Let R be an H1-ring, and $n \geq 2$. If $\tilde{L} = \{u_1,\dots,u_m\}$, $m \leq n$, is a free part of R^n such that the free module L, of basis $\tilde{L}$, is a subspace of R^n, then there exists a finite subset C of R^n such that $\tilde{L} \cup C$ is a basis of R^n.*

Proof. By definition there exists an R-module M such that $L \oplus M = R^n$. Thus M is stably free, and by [23, (2.11), pp. 292, 296], M is free. Taking C as a basis of M, we get the result. □

We now introduce the analog of the complexes of [19],

Definition 1.5. (*Normalized and generic complexes*).

(1) We define $C_t(n)$ as the free abelian group spanned by the t-tuples $(v_0,\dots,v_t)$ of points of $\mathbb{P}^n(R)$, subjected to the normalization condition $(v_0,\dots,v_t) = 0$, if $v_i = v_{i-1}$ for some i, $1 \leq i \leq t$, and such that: *if $v_{i_1},\dots,v_{i_k}$ are linearly independent in R^{n+1}, then they are a basis of a subspace of R^{n+1}.*

We can now construct the following complex augmented over $\mathbb{Z}$:

$$\cdots \to C_k(n) \xrightarrow{d_k} C_{k-1}(n) \to \cdots \to C_0(n) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $d_k(v_0,\dots,v_k) = \sum_{i=0}^{k} (-1)^i (v_0,\dots,\hat{v}_i,\dots,v_k)$. (with the usual notations) We call this complex $C_*(n)$.

(2) We define $C_*^{\text{gen},m}(n)$ as the subcomplex of $C_*(n)$ spanned by m-generic tuples.

Remark 1.6. In (1.5) the complex $C_*(n)$ is a quotient of the standard simplicial complex constructed over $\mathbb{P}^n(R)$.

The most interesting application of H1-rings is the following "crucial" lemma,

Lemma 1.7. *If $m \leq n$, the complex $C_*^{\mathrm{gen},m}(n)$ augmented over $\mathbb{Z}$ is acyclic.*

Proof. The proof is standard, we must show that every cycle is a boundary. Let $z \in C_k^{\mathrm{gen},m}(n)$, a cycle, then $z = \sum_{i \in I} n_i(v_{0,i}, \dots, v_{k,i})$, $n_i \in \mathbb{Z}$, $v_{i,j} \in \mathbb{P}^n(R)$ and $\mathrm{card}(I) < \infty$. We want $v \in \mathbb{P}^n(R)$ such that $(v, v_{0,i}, \dots, v_{k,i})$ are m-generic for all i. For each m-generic tuples $(v_{0,i}, \dots, v_{k,i})$, we construct all the m-generic configurations that we complete in hyperplanes of R^{n+1} (depends on whether $m < n$ or not). As R is H1, this finite family of hyperplanes has a common supplement of rank one, thus an element v of $\mathbb{P}^n(R)$, which, by construction, is m-generic with $(v_{0,i}, \dots, v_{k,i})$, and if $\tilde{z} = \sum_{i \in I} n_i(v, v_{0,i}, \dots, v_{k,i}) \in C_{k+1}^{\mathrm{gen},m}(n)$, then $d_{k+1}(\tilde{z}) = z$. □

1.1. Study of the associated spectral sequence

In the following, G denotes $GL_3(R)$ and C_* denotes $C_*(2)$. For group homology we refer to [7, Section VII.5, pp. 168–170]. Since C_* is acyclic, we have a (transposed) spectral sequence

$$E^1_{p,q} = \mathrm{H}_p(G; C_q) \Rightarrow \mathrm{H}_{p+q}(G; \mathbb{Z}).$$

Our goal is to show that we have an embedding $\mathrm{H}_3(GL_2(R); \mathbb{Q}) \hookrightarrow E^{\infty}_{3,0} \otimes_{\mathbb{Z}} \mathbb{Q}$, and deduce from this that we have an injection $\mathrm{H}_3(GL_2(R); \mathbb{Q}) \rightarrowtail \mathrm{H}_3(GL_3(R); \mathbb{Q})$, where the last map is induced by the stabilization morphism $GL_2(R) \rightarrowtail GL_3(R)$,

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

For this we investigate our spectral sequence in low degree, and we obtain the following results, which is the first step of our investigation. For computations of the differentials, and some complements on groups homology, we refer to the Appendix. First, take a look at $E^1_{p,0}$, and $E^1_{p,1}$.

1.1. Computation of $E^1_{p,0}$. We have $C_0 = \mathbb{Z}[G.s_{0,0}]$ with $s_{0,0} = ([e_1])$. Subsequently,

$$B_{0,0} = Stab_G(s_{0,0}) = \begin{pmatrix} R^\times & * \\ 0 & GL_2(R) \end{pmatrix}$$

thus $\mathrm{H}_*(B_{0,0}; \mathbb{Z}) \cong \mathrm{H}_*(R^\times \times GL_2(R); \mathbb{Z})$. As $C_0 = \mathrm{Ind}^G_{B_{0,0}} \mathbb{Z}$, it follows that $E^1_{p,0} \cong \mathrm{H}_p(R^\times \times GL_2(R); \mathbb{Z})$ and by construction $d^1_{p,0} = 0$.

1.2. Computation of $E^1_{p,1}$. We have $C_1 = \mathbb{Z}[G.s_{1,0}]$, where $s_{1,0} = ([e_1], [e_2])$.

$$B_{1,0} = Stab_G(s_{1,0}) = \begin{pmatrix} R^\times & 0 & * \\ 0 & R^\times & * \\ 0 & 0 & R^\times \end{pmatrix},$$

$$C_1 = \mathrm{Ind}^G_{B_{1,0}} \mathbb{Z},$$

and then,

$$E^1_{p,1} \cong \mathrm{H}_p(B_{1,0};\mathbb{Z}) \cong \mathrm{H}_p(R^\times \times R^\times \times R^\times;\mathbb{Z}).$$

Since $d_1(s_{1,0}) = ([e_2]) - ([e_1]) = (w-1).([e_1])$, with

$$w = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $B_{1,0} \leq B_{0,0}$. We are in the hypothesis of (A.1), and $w^{-1}(R^\times \times R^\times \times R^\times)w = R^\times \times R^\times \times R^\times$. Thus

$$d^1_{p,1} = \mathrm{cor}^{B_{0,0}}_{B_{1,0}} \circ (w^{-1} - 1). \qquad (1)$$

We can now state the first main result,

Proposition 1.8. *We have* $\mathrm{H}_3(GL_2(R);\mathbb{Q}) \hookrightarrow E^2_{3,0} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Proof. *The strategy*: We want to construct a surjective map (split by the induced morphism in homology by $GL_2(R) \rightarrowtail GL_3(R)$) from $\mathrm{H}_3(R^\times \times GL_2(R);\mathbb{Q})$ ($\cong E^1_{3,0} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$) to $\mathrm{H}_3(GL_2(R);\mathbb{Q})$ and show that its kernel contains $\mathrm{Im}(d^1_{3,1}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ ($= \mathrm{Im}(\mathrm{cor} \circ (w^{-1} - 1)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$), this will show that $\mathrm{H}_3(GL_2(R);\mathbb{Q})$ is a component of $E^2_{3,0} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Recall (cf. [7, (6.4)(ii), p. 123]) that if A is an abelian group, we have an isomorphism $\mathrm{H}_n(A;\mathbb{Q}) \cong \bigwedge^n_{\mathbb{Q}}(A)$. For $n=3$ (resp. $n=2$) this isomorphism is given by $a \wedge b \wedge c \mapsto \mathbf{c}(a,b,c)$ (resp. $a \wedge b \mapsto \mathbf{c}(a,b)$), where the symbol $\mathbf{c}(\,)$ is defined in Section A.2.1, and its inverse is induced by $[a|b|c] \otimes 1 \mapsto \frac{1}{6}(a \wedge b \wedge c)$ (resp. $[a|b] \otimes 1 \mapsto \frac{1}{2}(a \wedge b)$). Denote $A_i = R^\times$ for $i = 1,2,3$, this allows us to control the actions on the different components of the "torus" $R^\times \times R^\times \times R^\times$. By Künneth ([7, (5.8), p. 120] or [24, (6.1.13), p. 165]), applied to $(A_1 \times A_2) \times A_3$, we get

$$\mathrm{H}_3(A_1 \times A_2 \times A_3;\mathbb{Q}) \cong \bigwedge\nolimits^3_{\mathbb{Q}}(A_1 \times A_2) \oplus \bigwedge\nolimits^2_{\mathbb{Q}}(A_1 \times A_2) \otimes A_3 \oplus (A_1 \times A_2) \otimes \bigwedge\nolimits^2_{\mathbb{Q}}(A_3) \oplus \bigwedge\nolimits^3_{\mathbb{Q}}(A_3).$$

And, in the same way,

$$\mathrm{H}_3(R^\times \times GL_2(R);\mathbb{Q}) \cong \bigwedge\nolimits^3_{\mathbb{Q}}(R^\times) \oplus \bigwedge\nolimits^2_{\mathbb{Q}}(R^\times) \otimes R^\times \oplus R^\times \otimes \mathrm{H}_3(GL_2(R);\mathbb{Q}) \oplus \mathrm{H}_3(GL_2(R);\mathbb{Q}).$$

Our goal is to describe

$$\mathrm{cor} \circ (w^{-1} - 1) : \mathrm{H}_3(A_1 \times A_2 \times A_3;\mathbb{Q}) \to \mathrm{H}_3(R^\times \times GL_2(R);\mathbb{Q}).$$

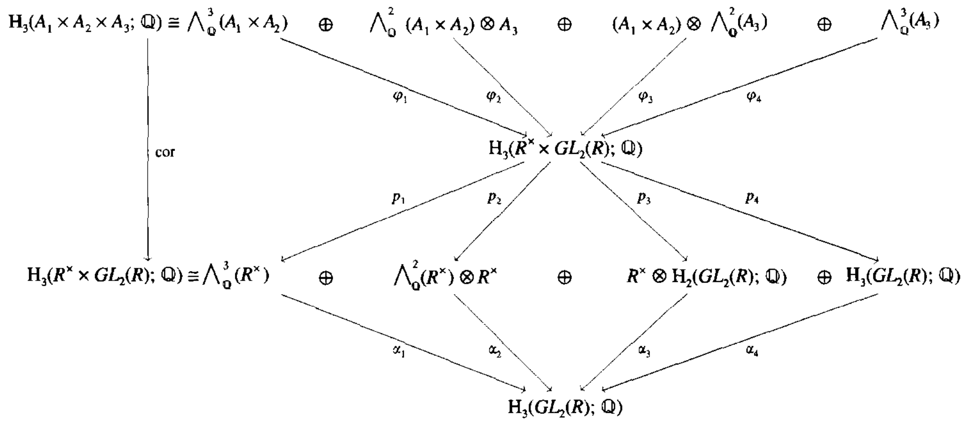

Fig. 1.

We have four projections:

$$p_1 : \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \to \bigwedge\nolimits^3_{\mathbb{Q}} (R^\times), \text{ induced by } \mathrm{pr}_1 : R^\times \times GL_2(R) \to R^\times,$$

$$p_2 : \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \to \bigwedge\nolimits^2_{\mathbb{Q}} (R^\times) \otimes R^\times, \text{ induced by}$$

$$(\text{class of } [(\alpha_1, g_1)|(\alpha_2, g_2)|(\alpha_3, g_3)] \otimes 1) \mapsto \tfrac{1}{2}(\alpha_1 \wedge \alpha_2) \otimes \det g_3,$$

$$p_3 : \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \to R^\times \otimes \mathrm{H}_3(GL_2(R); \mathbb{Q}) \text{ induced by}$$

$$(\text{class of } [(\alpha_1, g_1)|(\alpha_2, g_2)|(\alpha_3, g_3)] \otimes 1) \mapsto \alpha_1 \otimes (\text{class of } [g_2|g_3] \otimes 1),$$

$$p_4 : \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \to \mathrm{H}_3(GL_2(R); \mathbb{Q}), \text{ induced by}$$

$$\mathrm{pr}_2 : R^\times \times GL_2(R) \to GL_2(R).$$

We want to construct maps φ_1, φ_2, φ_3, φ_4 and α_1, α_2, α_3, α_4, where their relationships are shown in Fig. 1. We then have

$$\mathrm{cor} = (p_1 \oplus p_2 \oplus p_3 \oplus p_4) \circ (\varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4),$$

thus,

$$\mathrm{cor} \circ (w^{-1} - 1) = (p_1 \oplus p_2 \oplus p_3 \oplus p_4) \circ (\varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4) \circ (w^{-1} - 1).$$

Since A_3 represents the last component unaffected by the action of w^{-1}, we have

$$(w^{-1} - 1) \,|_{\bigwedge_{\mathbb{Q}}(A_3)} = 0.$$

Then it is just necessary to compute φ_1, φ_2 and φ_3, and the α_i. Begin by constructing α_i. We want to go from $\mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q})$ to $\mathrm{H}_3(GL_2(R); \mathbb{Q})$. Define the following maps:

$\alpha_1 : \bigwedge^3_{\mathbb{Q}}(R^\times) \to \mathrm{H}_3(GL_2(R);\mathbb{Q})$ induced by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, split by $\det : GL_2(R) \to R^\times$;

$$\begin{array}{ccc} \bigwedge^2_{\mathbb{Q}}(R^\times) \otimes R^\times & \xrightarrow{\alpha_2} & \mathrm{H}_3(GL_2(R);\mathbb{Q}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_3(R^\times \times R^\times;\mathbb{Q}) & \end{array}$$

is given by the factorisation

$$\begin{array}{ccc} (a \wedge b) \otimes c & \xmapsto{\alpha_2} & c\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)\right) \\ \Big\updownarrow \cong & & \Big\uparrow \\ c(a,b) \otimes c & \longmapsto & c((a,1),(b,1),(1,c))\,. \end{array}$$

The map, $R^\times \otimes \mathrm{H}_2(GL_2(R);\mathbb{Q}) \xrightarrow{\alpha_3} \mathrm{H}_3(GL_3(R);\mathbb{Q})$ is constructed as follows:

(1) First, we have α_3' given by

$$\begin{array}{ccc} R^\times \otimes \bigwedge^2_{\mathbb{Q}}(R^\times) & \xrightarrow{\alpha_3'} & \mathrm{H}_3(GL_2(R);\mathbb{Q}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_3(R^\times \times R^\times;\mathbb{Q}) & \end{array}$$

$$\alpha_3'(a \otimes (b \wedge c)) = \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(2) We also have the map α_3'' given by

$$\begin{array}{ccc} R^\times \otimes K_2(R)_{\mathbb{Q}} & \xrightarrow{\alpha_3''} & \mathrm{H}_3(GL_2(R);\mathbb{Q}) \\ & {\scriptstyle (1)}\searrow \quad \nearrow{\scriptstyle (2)} & \\ & \mathrm{H}_3(R^\times \times GL_2(R);\mathbb{Q}) & \end{array}$$

In the above diagram (1) is given by the shuffle product, because $\mathrm{K}_2(R) \hookrightarrow \mathrm{H}_2(GL_2(R);\mathbb{Z})$. But (2) is induced by the "product" map $R^\times \times GL_2(R) \to GL_2(R)$, $(\lambda, g) \mapsto \lambda.g$.

Explicitly,

$$\alpha_3''(a\otimes\{b,c\}) = \mathbf{c}\left(\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & c^{-1}\end{pmatrix}\right).$$

Remark 1.9. As we know $K_2(R)_{\mathbb{Q}}$ is identified with $\overline{\mathrm{H}}_2(SL_2(R);\mathbb{Q})$, where $\overline{\mathrm{H}}_2(SL_2(R);\mathbb{Q})$ is a useful notation for $\mathrm{H}_0(R^\times;\mathrm{H}_2(SL_2(R);\mathbb{Q}))$, and in fact $\overline{\mathrm{H}}_2(SL_2(R);\mathbb{Q}) \hookrightarrow \mathrm{H}_2(GL_2(R);\mathbb{Q})$ by (A.8(1)). Furthermore by (A.7) we have

$$R^\times \otimes \overline{\mathrm{H}}_2(SL_2(R);\mathbb{Q}) \hookrightarrow \mathrm{H}_3(GL_2(R);\mathbb{Q}),$$

where the maps come from $R^\times \times SL_2(R) \to GL_2(R)$, $(\lambda, g) \mapsto \lambda.g$ (cf. (A.7)).

Then define α_3 as follows:

$$\alpha_3(a\otimes(b\wedge c) + a'\otimes\{b',c'\}) = \alpha_3'(a\otimes(b\wedge c)) + \tfrac{1}{2}\alpha_3''(a'\otimes\{b',c'\}).$$

Lemma 1.10. *For all $a,b,c\in R^\times$, set $z = a\otimes(b\wedge c) - b\otimes(c\wedge a) + a\otimes\{c,b\} - b\otimes\{a,c\}$. Then $\alpha_3(z)=0$.*

Proof. Observe first that if $\lambda\in R^\times$, $g,g'\in GL_2(R)$ are such that,

$$g, g', \begin{pmatrix} \lambda & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & \lambda\end{pmatrix}$$

pairwise commute, then,

$$\mathbf{c}\left(\begin{pmatrix} 1 & 0\\ 0 & \lambda\end{pmatrix}, g, g'\right) = \frac{1}{2}\mathbf{c}\left(\begin{pmatrix} \lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}, g, g'\right) + \frac{1}{2}\mathbf{c}\left(\begin{pmatrix} \lambda & 0\\ 0 & \lambda\end{pmatrix}, g, g'\right), \tag{2}$$

$$\mathbf{c}\left(\begin{pmatrix} \lambda & 0\\ 0 & 1\end{pmatrix}, g, g'\right) = \frac{1}{2}\mathbf{c}\left(\begin{pmatrix} \lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, g, g'\right) + \frac{1}{2}\mathbf{c}\left(\begin{pmatrix} \lambda & 0\\ 0 & \lambda\end{pmatrix}, g, g'\right). \tag{3}$$

Moreover, for all $a,b,c\in R^\times$,

$$\mathbf{c}\left(\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} b & 0\\ 0 & b^{-1}\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & c^{-1}\end{pmatrix}\right) = 0. \tag{4}$$

We have

$$\begin{aligned}
&\alpha_3(a\otimes(b\wedge c) - b\otimes(c\wedge a))\\
&= \mathbf{c}\left(\begin{pmatrix} 1 & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix}\right) - \mathbf{c}\left(\begin{pmatrix} 1 & 0\\ 0 & b\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right)\\
&= \mathbf{c}\left(\begin{pmatrix} 1 & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix}\right) - \mathbf{c}\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & b\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix}\right).
\end{aligned}$$

Let

$$z' = \mathbf{c}\left(\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & 1\end{pmatrix}\right) - \mathbf{c}\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & 1\end{pmatrix}\right),$$

then, using the relations above, we have

$$\begin{aligned} z' = &\frac{1}{4}\mathbf{c}\left(\begin{pmatrix}a^{-1} & a\\ 0 & a\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\right)\\ &+\frac{1}{4}\mathbf{c}\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & b^{-1}\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\right). \end{aligned}$$

In the same way, we see that

$$\begin{aligned} &\alpha_3(a\otimes\{c,b\} - b\otimes\{a,c\})\\ &= -\frac{1}{4}\mathbf{c}\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & b^{-1}\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\right)\\ &\quad -\frac{1}{4}\mathbf{c}\left(\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}, \begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\right), \end{aligned}$$

and by adding to z', we get $\alpha_3(z) = 0$. □

Note also that

$$\begin{aligned} \alpha_2((a\wedge b)\otimes c) &= \mathbf{c}\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & c\end{pmatrix}\right)\\ &= \mathbf{c}\left(\begin{pmatrix}1 & 0\\ 0 & c\end{pmatrix}, \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}\right)\\ &= \alpha_3(c\otimes(a\wedge b)). \end{aligned}$$

And finally set $\alpha_4 = 1_{\mathrm{H}_3(GL_2(R);\mathbb{Q})}$. Recall that

$$w^{-1} = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad w^{-1}\begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & 1\end{pmatrix} w = \begin{pmatrix}b & 0 & 0\\ 0 & a & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Denote,

$$\sigma_1 = (w^{-1} - 1)|_{\bigwedge^3_{\mathbb{Q}}(A_1 \times A_2)}, \qquad \sigma_2 = (w^{-1} - 1)|_{\bigwedge^2_{\mathbb{Q}}(A_1 \times A_2) \otimes (A_3)} \quad \text{and}$$
$$\sigma_3 = (w^{-1} - 1)|_{(A_1 \times A_2) \otimes \bigwedge^2_{\mathbb{Q}}(A_3)}.$$

We have,

$$\sigma_1 : \bigwedge\nolimits^3_{\mathbb{Q}}(A_1 \times A_2) \to \bigwedge\nolimits^3_{\mathbb{Q}}(A_1 \times A_2)$$
$$(a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3) \mapsto (b_1,a_1) \wedge (b_2,a_2) \wedge (b_3,a_3) - (a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)$$

$$\sigma_2 : \bigwedge\nolimits^2_{\mathbb{Q}}(A_1 \times A_2) \otimes A_3 \to \bigwedge\nolimits^2_{\mathbb{Q}}(A_1 \times A_2) \otimes A_3$$
$$((a,b) \wedge (c,d)) \otimes e \mapsto ((b,a) \wedge (d,c)) \otimes e - ((a,b) \wedge (c,d)) \otimes e$$

$$\sigma_3 : (A_1 \times A_2) \otimes \bigwedge\nolimits^2_{\mathbb{Q}}(A_3) \to (A_1 \times A_2) \otimes \bigwedge\nolimits^2_{\mathbb{Q}}(A_3)$$
$$(a,b) \otimes (c \wedge d) \mapsto (b,a) \otimes (c \wedge d) - (a,b) \otimes (c \wedge d).$$

We now construct the maps φ_l. We have

$$\begin{array}{ccc} \bigwedge^3_{\mathbb{Q}}(A_1 \times A_2) & \xrightarrow{\quad \varphi_1 \quad} & \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \\ & \searrow \qquad \nearrow & \\ & \mathrm{H}_3(A_1 \times A_2; \mathbb{Q}) & \end{array}$$

the second map is induced by $A_1 \times A_2 \to R^\times \times GL_2(R)$,

$$(a,b) \mapsto \left(a, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Explicitly we get,

$$\varphi_1((a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)) = \mathbf{c}\left(\left(a_1, \begin{pmatrix} b_1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(a_2, \begin{pmatrix} b_2 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(a_3, \begin{pmatrix} b_3 & 0 \\ 0 & 1 \end{pmatrix}\right)\right)$$

and then,

$$\begin{aligned}
p_1 \circ \varphi_1((a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)) &= a_1 \wedge a_2 \wedge a_3,\\
p_2 \circ \varphi_1((a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)) &= \tfrac{1}{2}((a_1 \wedge a_2) \otimes b_3 - (a_2 \wedge a_1) \otimes b_3\\
&\quad + (a_2 \wedge a_3) \otimes b_1 - (a_1 \wedge a_3) \otimes b_2\\
&\quad + (a_3 \wedge a_1) \otimes b_2 - (a_3 \wedge a_2) \otimes b_1)\\
&= (a_1 \wedge a_2) \otimes b_3 + (a_2 \wedge a_3) \otimes b_1\\
&\quad - (a_1 \wedge a_3) \otimes b_2,\\
p_3 \circ \varphi_1((a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)) &= a_1 \otimes \mathbf{c}\left(\begin{pmatrix} b_2 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} b_3 & 0\\ 0 & 1\end{pmatrix}\right)\\
&\quad -a_2 \otimes \mathbf{c}\left(\begin{pmatrix} b_1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} b_3 & 0\\ 0 & 1\end{pmatrix}\right)\\
&\quad +a_3 \otimes \mathbf{c}\left(\begin{pmatrix} b_1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix} b_2 & 0\\ 0 & 1\end{pmatrix}\right)\\
&= a_1 \otimes (b_2 \wedge b_3) - a_2 \otimes (b_1 \wedge b_3)\\
&\quad + a_3 \otimes (b_1 \wedge b_2),\\
p_4 \circ \varphi_1((a_1,b_1) \wedge (a_2,b_2) \wedge (a_3,b_3)) &= \alpha_1(b_1 \wedge b_2 \wedge b_3).
\end{aligned}$$

We have

$$\begin{array}{ccc}
\bigwedge^2_{\mathbb{Q}}(A_1 \times A_2) \otimes A_3 & \xrightarrow{\quad \varphi_2 \quad} & \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q})\\
 & {\scriptstyle (1)} \searrow \qquad \nearrow {\scriptstyle (2)} & \\
 & \mathrm{H}_3(A_1 \times A_2 \times A_3; \mathbb{Q}) &
\end{array}$$

the arrow (1) is given by the shuffle product, besides (2) is induced by

$$(a,b,c) \mapsto \left(a, \begin{pmatrix} b & 0\\ 0 & c\end{pmatrix}\right).$$

Explicitly,

$$\varphi_2(((a,b) \wedge (c,d)) \otimes e) = \mathbf{c}\left(\left(a, \begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}\right), \left(c, \begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 0\\ 0 & e\end{pmatrix}\right)\right).$$

We then have

$$\begin{aligned}
p_1 \circ \varphi_2(((a,b) \wedge (c,d)) \otimes e) &= 0,\\
p_2 \circ \varphi_2(((a,b) \wedge (c,d)) \otimes e) &= (a \wedge c) \otimes e,
\end{aligned}$$

$$
\begin{aligned}
p_3 \circ \varphi_2(((a,b)\wedge(c,d))\otimes e) &= a \otimes \mathbf{c}\left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\right) \\
&\quad + c \otimes \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= a \otimes (d \wedge e) + a \otimes \{e,d\} + c \otimes (e \wedge b) + c \otimes \{b,e\}, \\
p_4 \circ \varphi_2(((a,b)\wedge(c,d))\otimes e) &= \alpha_2((b \wedge d)\otimes e).
\end{aligned}
$$

We have

$$
\begin{array}{ccc}
(A_1 \times A_2) \otimes \bigwedge^2_{\mathbb{Q}}(A_3) & \xrightarrow{\varphi_3} & \mathrm{H}_3(R^\times \times GL_2(R); \mathbb{Q}) \\
& \searrow \qquad \nearrow & \\
& \mathrm{H}_3(A_1 \times A_2 \times A_3; \mathbb{Q}) &
\end{array}
$$

explicitly,

$$
\varphi_3((a,b)\otimes(c\wedge d)) = \mathbf{c}\left(\left(a, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right)\right).
$$

And by combining with the projections, we get

$$
\begin{aligned}
p_1 \circ \varphi_3((a,b)\otimes(c\wedge d)) &= 0, \\
p_2 \circ \varphi_3((a,b)\otimes(c\wedge d)) &= 0, \\
p_3 \circ \varphi_3((a,b)\otimes(c\wedge d)) &= a \otimes \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right) \\
&= a \otimes (c \wedge d), \\
p_4 \circ \varphi_3((a,b)\otimes(c\wedge d)) &= \alpha_3((a,b)\otimes(c\wedge d)).
\end{aligned}
$$

Let $\pi = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \mathrm{id}$, and $x = ((a_1,b_1)\wedge(a_2,b_2)\wedge(a_3,b_3), ((c_1,d_1)\wedge(c_2,d_2))\otimes c_3, (e_1,f_1)\otimes(e_2\wedge e_3), z) \in \mathrm{H}_3(R^\times \times R^\times \times R^\times; \mathbb{Q})$

$$
\begin{aligned}
(w^{-1}-1)(x) = (&(b_1,a_1)\wedge(b_2,a_2)\wedge(b_3,a_3) - (a_1,b_1)\wedge(a_2,b_2)\wedge(a_3,b_3), \\
&((d_1,c_1)\wedge(d_2,c_2))\otimes c_3 - ((c_1,d_1)\wedge(c_2,d_2))\otimes c_3, \\
&(f_1,e_1)\otimes(e_2\wedge e_3) - (e_1,f_1)\otimes(e_2\wedge e_3), 0).
\end{aligned}
$$

Denote $(w^{-1}-1)(x)$ by X, then a routine (if somewhat tedious) calculation shows that $\pi(\mathrm{cor}(X)) = 0$. In other words, $\mathrm{Ker}(\pi) \supset \mathrm{Im}(d^1_{3,1})$ and π is a split surjection, thus $\mathrm{Im}(d^1_{3,1}) \cap \mathrm{Im}(\mathrm{H}_3(GL_2(R); \mathbb{Q})) = 0$, and this ends the proof of Proposition 1.8. □

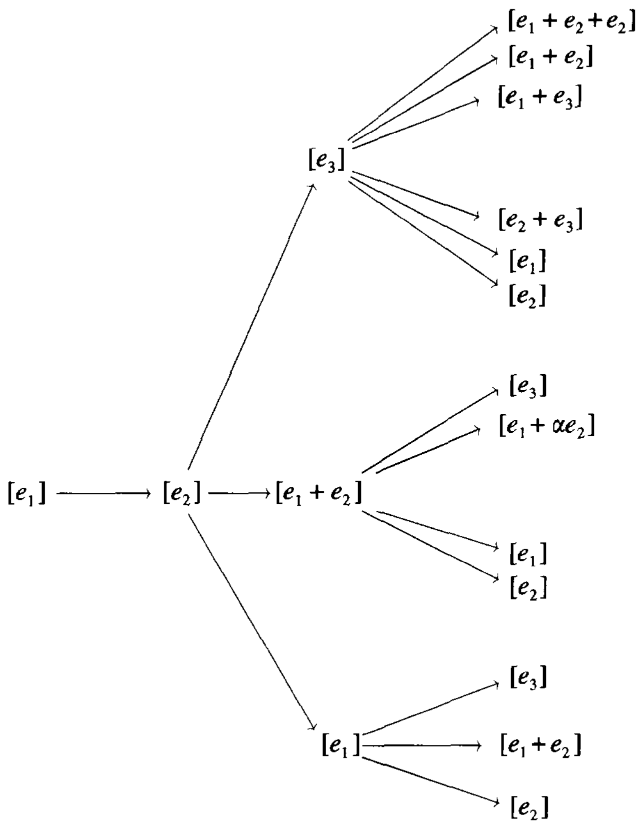

Fig. 2.

1.3. Proof of $E^{\infty}_{3,0} = E^{2}_{3,0}$. It is necessary to divide the proof in technical steps, the global strategy is close to Sah [19], and is as follows: *the first differential ending in* $E^2_{3,0}$ *is* $d^2_{2,2}$. *Proposition* 1.20 *shows that* $E^2_{2,2} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{3}] = 0$ *and then* $d^2_{2,2} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. *It follows that* $E^2_{3,0} \otimes_{\mathbb{Z}} \mathbb{Q} \cong E^3_{3,0} \otimes_{\mathbb{Z}} \mathbb{Q}$. *The differential abutting to* $E^3_{3,0}$ *is* $d^3_{1,3}$. *We prove in* (1.19(2)), *that* $E^2_{1,3} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] = 0$. *Then* $E^3_{1,3} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, $d^3_{1,3} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ *and subsequently* $E^2_{3,0} \otimes_{\mathbb{Z}} \mathbb{Q} \cong E^4_{3,0}$. *As the spectral sequence is of the first quadrant, the last differential involved in the computation of* $E^{\infty}_{3,0}$ *is* $d^4_{0,4}$. *We show, once again, that* $E^2_{0,4} = 0$, *thus* $E^4_{0,4} = 0$ *and then* $d^4_{0,4} = 0$. *The crucial step in this last result is Proposition* 1.13.

For the good understanding of the different n-cells ($0 \le n \le 3$), we give the diagram shown in Fig. 2, in which each oriented path shows a "representative" n-cell ($0 \le n \le 3$), and α is an element of $R^{\times} - \{1\}$.

First, we need a description of the terms $E^1_{*,2}$, and $E^1_{*,3}$, of the spectral sequence. In the following, we will use implicitly (1.4), the "Eckmann–Shapiro lemma", the Proposition A.2, and the following theorem (see also [13, (2.2.2), p. 40]):

Theorem 1.11. *Let R be an $S(\infty)$-ring, and G_1 be a subgroup of $GL_p(R)$ and G_2 be a subgroup of $GL_q(R)$, assume that G_1 or G_2 contains the group of scalar matrices. Let M be a submodule of $M_{p,q}(R)$ such that $G_1M = M = MG_2$, then the imbedding*

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \hookrightarrow \begin{pmatrix} G_1 & M \\ 0 & G_2 \end{pmatrix}$$

induces an isomorphism in homology with coefficients in $\mathbb{Z}$.

Remark 1.12. For the proof of (1.11), as R is $S(\infty)$, we use Proposition 1.10 of [17, p. 125], and next we rewrite, word for word, the proof of Theorem 1.9 of [22, p. 211].

1.3.1. Computation of $E^1_{p,2}$. We have

$$C_2 = \bigoplus_{i=0}^{2} \mathbb{Z}[G.s_{2,i}]$$

where

$$s_{2,0} = ([e_1],[e_2],[e_3]),$$

$$s_{2,1} = ([e_1],[e_2],[e_1+e_2]),$$

$$s_{2,2} = ([e_1],[e_2],[e_1]),$$

$$B_{2,i} = Stab_G(s_{2,i}).$$

Then

$$B_{2,0} = \begin{pmatrix} R^\times & 0 & 0 \\ 0 & R^\times & 0 \\ 0 & 0 & R^\times \end{pmatrix},$$

$$B_{2,1} = \left\{ \begin{pmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & \mu \end{pmatrix} \in G, \lambda, \mu \in R^\times \right\},$$

$$B_{2,2} = \begin{pmatrix} R^\times & 0 & * \\ 0 & R^\times & * \\ 0 & 0 & R^\times \end{pmatrix}.$$

Thus,

$$C_2 = \bigoplus_{i=0}^{2} \mathrm{Ind}_{B_{2,i}}^{B_{1,0}} \mathbb{Z},$$

$$E^1_{p,2} \cong \bigoplus_{i=0}^{2} \mathrm{H}_p(B_{2,i}; \mathbb{Z})$$

$$\cong \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{2,0} \oplus \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{2,1} \oplus \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{2,2},$$

where the indexes indicate of what type of cells the group homology come from. The computation of the differential is as follows:

$$d_2(s_{2,0}) = ([e_2],[e_3]) - ([e_1],[e_3]) + ([e_1],[e_2]) = (\tau_1^{2,0} - \tau_2^{2,0} + 1).s_{1,0},$$

with

$$\tau_1^{2,0} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (\tau_1^{2,0})^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tau_2^{2,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (\tau_2^{2,0})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then, for $z_0 \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{2,0}$, we have,

$$d^1_{p,2}(z_0) = \mathrm{cor}_{B_{2,0}}^{B_{1,0}} \circ ((\tau_1^{2,0})^{-1} - (\tau_2^{2,0})^{-1} + 1) z_0.$$

Let

$$d_2(s_{2,1}) = ([e_2],[e_1+e_2]) - ([e_1],[e_1+e_2]) + ([e_1],[e_2]) = (\tau_1^{2,1} - \tau_2^{2,1} + 1).s_{1,0},$$

with

$$\tau_1^{2,1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (\tau_1^{2,1})^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau_2^{2,1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (\tau_2^{2,1})^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $z_1 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{2,1}$ then,

$$d^1_{p,2}(z_1) = \mathrm{cor}_{B_{2,1}}^{B_{1,0}} \circ ((\tau_1^{2,1})^{-1} - (\tau_2^{2,1})^{-1} + 1)z_1.$$

As $\tau_1^{2,1}, \tau_2^{2,1}$ act trivially on $\mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{2,1}$, we see that $d^1_{p,2}(z_1) = z_1$. Next,

$$d_2(s_{2,2}) = ([e_2],[e_1]) + ([e_1],[e_2]) = (\tau_1^{2,2} + 1).s_{1,0},$$

with

$$\tau_1^{2,2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = w, \qquad (\tau_1^{2,2})^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $z_2 \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{2,2}$ then,

$$d^1_{p,2}(z_2) = \mathrm{cor}_{B_{2,2}}^{B_{1,0}} \circ ((\tau_1^{2,2})^{-1} + 1)z_2.$$

And this gives a complete description of $d^1_{*,2}$.

1.3.2. Computation of $E^1_{p,3}$. See Fig. 2 for the representatives of the 3-cells. We have,

$$C_3 = \bigoplus_{i=0}^{11} \mathbb{Z}[G.s_{3,i}] \oplus \left(\bigoplus_{\alpha \in R^\times - \{1\}} \mathbb{Z}[G.s^\alpha_{3,12}] \right)$$

with

$$\begin{aligned}
s_{3,0} &= ([e_1],[e_2],[e_3],[e_1+e_2+e_3]),\\
s_{3,1} &= ([e_1],[e_2],[e_3],[e_1+e_2]),\\
s_{3,2} &= ([e_1],[e_2],[e_3],[e_2+e_3]),\\
s_{3,3} &= ([e_1],[e_2],[e_3],[e_1+e_3]),\\
s_{3,4} &= ([e_1],[e_2],[e_3],[e_1]),\\
s_{3,5} &= ([e_1],[e_2],[e_3],[e_2]),\\
s_{3,6} &= ([e_1],[e_2],[e_1+e_2],[e_3]),\\
s_{3,7} &= ([e_1],[e_2],[e_1+e_2],[e_1]),\\
s_{3,8} &= ([e_1],[e_2],[e_1+e_2],[e_2]),\\
s_{3,9} &= ([e_1],[e_2],[e_1],[e_3]),\\
s_{3,10} &= ([e_1],[e_2],[e_1],[e_1+e_2]),\\
s_{3,11} &= ([e_1],[e_2],[e_1],[e_2]),\\
s^\alpha_{3,12} &= ([e_1],[e_2],[e_1+e_2],[e_1+\alpha e_2]).
\end{aligned}$$

Denote by $B_{3,i}$ the stabilizer of $s_{3,i}$ in G, $0 \le i \le 11$, and by $B^{\alpha}_{3,12}$ those of $s^{\alpha}_{3,12}$. We then have,

$$B_{3,0} = R^{\times}.I_3,\ B_{3,1} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \in G,\ \lambda,\mu \in R^{\times} \right\} = B_{3,6},$$

$$B_{3,2} = \left\{ \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \in G,\ \lambda,\mu \in R^{\times} \right\},\quad B_{3,3} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix} \in G,\ \lambda,\mu \in R^{\times} \right\},$$

$$B_{3,4} = B_{3,5} = B_{3,9} \cong R^{\times} \times R^{\times} \times R^{\times},$$

$$B_{3,7} = \left\{ \begin{pmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & \mu \end{pmatrix} \in G,\ \lambda,\mu \in R^{\times} \right\} = B_{3,8} = B_{3,10} = B^{\alpha}_{3,12},$$

$$B_{3,11} = \left\{ \begin{pmatrix} a & 0 & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \in G,\ a,b,c \in R^{\times} \right\}.$$

Hence,

$$\begin{aligned} E^1_{p,3} \cong\ & \mathrm{H}_p(R^{\times};\mathbb{Z})_{3,0} \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,1} \\ & \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,2} \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,3} \\ & \oplus \mathrm{H}_p(R^{\times}\times R^{\times}\times R^{\times};\mathbb{Z})_{3,4} \oplus \mathrm{H}_p(R^{\times}\times R^{\times}\times R^{\times};\mathbb{Z})_{3,5} \\ & \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,6} \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,7} \\ & \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,8} \oplus \mathrm{H}_p(R^{\times}\times R^{\times}\times R^{\times};\mathbb{Z})_{3,9} \\ & \oplus \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z})_{3,10} \oplus \mathrm{H}_p(R^{\times}\times R^{\times}\times R^{\times};\mathbb{Z})_{3,11} \\ & \oplus \left(\bigoplus_{\alpha \in R^{\times} - \{1\}} \mathrm{H}_p(R^{\times}\times R^{\times};\mathbb{Z}).\langle \alpha \rangle \right)_{3,12}. \end{aligned}$$

For the differentials, we get the following,

$$d_3(s_{3,0}) = (\tau_1^{3,0} - \tau_2^{3,0} + \tau_3^{3,0} - 1).s_{2,0},$$

as $R^{\times}$ is central in G, the action is trivial, and if $z_0 \in \mathrm{H}_p(R^{\times};\mathbb{Z})_{3,0}$, we deduce that $d^1_{p,3}(z_0) = 0$.

Let $d_3(s_{3,1}) = (\tau_1^{3,1} - \tau_2^{3,1} - 1).s_{2,0} + s_{2,1}$, with

$$\tau_1^{3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (\tau_1^{3,1})^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau_2^{3,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad (\tau_2^{3,1})^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If $z_1 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,1}$, then

$$d_{p,3}^1(z_1) = \mathrm{cor}_{B_{3,1}}^{B_{2,0}} \circ ((\tau_1^{3,1})^{-1} - (\tau_2^{3,1})^{-1} - 1)z_1 \oplus \mathrm{cor}_{B_{3,1}}^{B_{2,1}} z_1.$$

We have

$$d_3(s_{3,2}) = (-\tau_1^{3,2} + \tau_2^{3,2} - 1).s_{2,0} + \tau_3^{3,2} s_{2,1},$$

where

$$\tau_1^{3,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad (\tau_1^{3,2})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\tau_2^{3,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (\tau_2^{3,2})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau_3^{3,2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (\tau_3^{3,2})^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If $z_2 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,2}$, then

$$d_{p,3}^1(z_2) = \mathrm{cor}_{B_{3,2}}^{B_{2,0}} \circ (-(\tau_1^{3,2})^{-1} + (\tau_2^{3,2})^{-1} - 1)z_2 \oplus \mathrm{cor}_{B_{3,3}}^{B_{2,1}} \circ (\tau_3^{3,2})^{-1} z_2.$$

As $\tau_1^{3,2}$ and $\tau_2^{3,2}$ acts trivially on $\mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,2}$, we deduce $d_{p,3}^1(z_2) = \mathrm{cor}_{B_{3,2}}^{B_{2,0}}(-z_2) \oplus \mathrm{cor}_{B_{3,3}}^{B_{2,1}} \circ (\tau_3^{3,2})^{-1} z_2$. We have $d_3(s_{3,3}) = (\tau_1^{3,3} + \tau_2^{3,3} - 1).s_{2,0} - \tau_3^{3,3} s_{2,1}$, where

$$\tau_1^{3,3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad (\tau_1^{3,3})^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau_2^{3,3} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (\tau_2^{3,3})^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tau_3^{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (\tau_3^{3,3})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

If $z_3 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,3}$, then

$$d^1_{p,3}(z_3) = \mathrm{cor}^{B_{2,0}}_{B_{3,3}} \circ ((\tau_1^{3,3})^{-1} + (\tau_2^{3,3})^{-1} - 1) z_3 \oplus \mathrm{cor}^{B_{2,1}}_{B_{3,3}} \circ (-(\tau_3^{3,3})^{-1}) z_3.$$

Let

$$d_3(s_{3,4}) = (\tau_1^{3,4} - 1).s_{2,0} + (-\tau_2^{3,4} + 1).s_{2,2},$$

where $\tau_1^{3,4} = \tau_1^{2,0}$, $\tau_2^{3,4} = \tau_2^{2,0}$. If $z_4 \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{3,4}$, then

$$d^1_{p,3}(z_4) = \mathrm{cor}^{B_{2,0}}_{B_{3,4}} \circ ((\tau_1^{3,4})^{-1} - 1) z_4 \oplus \mathrm{cor}^{B_{2,2}}_{B_{3,4}} \circ (-(\tau_2^{3,4})^{-1} + 1) z_4.$$

Let

$$d_3(s_{3,5}) = (-\tau_1^{3,5} - 1).s_{2,0} + \tau_2^{3,5} s_{2,2},$$

where $\tau_1^{3,5} = \tau_2^{2,0}$, $\tau_2^{3,5} = \tau_1^{2,0}$. If $z_5 \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{3,5}$, then

$$d^1_{p,3}(z_5) = \mathrm{cor}^{B_{2,0}}_{B_{3,5}} \circ (-(\tau_1^{3,5})^{-1} - 1) z_5 \oplus \mathrm{cor}^{B_{2,2}}_{B_{3,5}} \circ (\tau_2^{3,5})^{-1} z_5.$$

Let $d_3(s_{3,6}) = (\tau_1^{3,6} - \tau_2^{3,6} + 1).s_{2,0} - s_{2,2}$, where $\tau_1^{3,6} = \tau_1^{2,1}$, $\tau_2^{3,6} = \tau_2^{2,1}$. If $z_6 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,6}$, then

$$d^1_{p,3}(z_6) = \mathrm{cor}^{B_{2,0}}_{B_{3,6}} \circ ((\tau_1^{3,6})^{-1} - (\tau_2^{3,6})^{-1} + 1) z_6 \oplus \mathrm{cor}^{B_{2,2}}_{B_{3,6}} (-z_6).$$

As $\tau_1^{3,6}$ and $\tau_2^{3,6}$ act trivially on $\mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,6}$, we deduce that

$$d^1_{p,3}(z_6) = \mathrm{cor}^{B_{2,0}}_{B_{3,6}} (z_6) \oplus \mathrm{cor}^{B_{2,2}}_{B_{3,6}} (-z_6).$$

We have

$$d_3(s_{3,7}) = (\tau_1^{3,7} - 1).s_{2,1} + (-\tau_2^{3,7} + 1).s_{2,2},$$

where $\tau_1^{3,7} = \tau_1^{2,1}$, $\tau_2^{3,7} = \tau_2^{2,1}$. If $z_7 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,7}$, then

$$d^1_{p,3}(z_7) = \mathrm{cor}^{B_{2,1}}_{B_{3,7}} \circ ((\tau_1^{3,7})^{-1} - 1) z_7 \oplus \mathrm{cor}^{B_{2,2}}_{B_{3,7}} \circ (-(\tau_2^{3,7})^{-1} + 1) z_7.$$

Let

$$d_3(s_{3,8}) = (-\tau_1^{3,8} - 1).s_{2,1} + \tau_2^{3,8} s_{2,2},$$

where

$$\tau_1^{3,8} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (\tau_1^{3,8})^{-1}, \qquad \tau_2^{3,8} = \tau_1^{2,1}.$$

If $z_8 \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,8}$, then

$$d^1_{p,3}(z_8) = \mathrm{cor}_{B_{3,8}}^{B_{2,1}} \circ (-(\tau_1^{3,8})^{-1} - 1)z_8 \oplus \mathrm{cor}_{B_{3,8}}^{B_{2,2}} \circ (\tau_2^{3,8})^{-1} z_8.$$

We have

$$d_3(s_{3,9}) = (w+1).s_{2,0} - s_{2,2}.$$

If $z_9 \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{3,9}$, then

$$d^1_{p,3}(z_9) = \mathrm{cor}_{B_{3,8}}^{B_{2,0}} \circ (w^{-1} + 1)z_9 \oplus \mathrm{cor}_{B_{3,9}}^{B_{2,2}}(-z_9).$$

Let

$$d_3(s_{3,10}) = (w'^{-1} + 1).s_{2,0} - s_{2,2},$$

where

$$w' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If $z_{10} \in \mathrm{H}_p(R^\times \times R^\times; \mathbb{Z})_{3,10}$, then

$$d^1_{p,3}(z_{10}) = \mathrm{cor}_{B_{3,10}}^{B_{2,0}} \circ (W' + 1)z_{10} \oplus \mathrm{cor}_{B_{3,10}}^{B_{2,2}}(-z_{10}).$$

We have

$$d_3(s_{3,11}) = (w^{-1} - 1).s_{2,2},$$

and if $z_{11} \in \mathrm{H}_p(R^\times \times R^\times \times R^\times; \mathbb{Z})_{3,11}$, then

$$d^1_{p,3}(z_{11}) = \mathrm{cor}_{B_{3,11}}^{B_{2,2}} \circ (w - 1)z_{11}.$$

$$d_3(s^\alpha_{3,12}) = (\tau_1^\alpha - \tau_2^\alpha + \tau_3^\alpha - 1).s_{2,1},$$

where τ_i^α are matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & \lambda \end{pmatrix} \quad (\text{with } g \in GL_2(R),\ \lambda \in R^\times).$$

As

$$B^\alpha_{3,12} = \begin{pmatrix} R^\times I_2 & * \\ 0 & R^\times \end{pmatrix},$$

the action of $(\tau_i^\alpha)^{-1}$ is trivial, then $d^1_{p,3}(z^\alpha_{12})=0$, for all $z^\alpha_{12}\in \mathrm{H}_p(R^\times\times R^\times;\mathbb{Z}).\langle\alpha\rangle$. And this gives a complete description of $d^1_{*,3}$.

Now, we can prove the first step,

1.3.3. 4-acyclicity of $E^1_{0,q}$ for $1\le q\le 4$. Recall that we denote the class of $v\in R^{n+1}$ in $\mathbb{P}^n(R)$ by $[v]$. The following result is the analogue of [19, pp. 293–295], with more explicit description and different methods

Proposition 1.13. *If R is* H1, *then $C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\cong E^1_{0,*}$ is 4-acyclic.*

Proof. Consider the following exact sequence of complexes of $\mathbb{Z}G$-modules $0\to C^{\mathrm{gen}}_*(n)\xrightarrow{i} C_*(n)\to Q_*(n)\to 0$, where $Q_*(n)=C_*(n)/C^{\mathrm{gen}}_*(n)$. For showing that $C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}$ is 4-acyclic, it is sufficient to prove that

(1) The sequence $0\to C^{\mathrm{gen}}_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\to C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\to Q_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\to 0$ is exact.

(2) $Q_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}$ is 4-acyclic.

(3) The map induced in homology by $C^{\mathrm{gen}}_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\to C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}$ is 0 in degree lower than 4.

Indeed, applying the homology long exact sequence to (1), we get

$$\cdots\to H_i\left(C^{\mathrm{gen}}_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\right)\to H_i\left(C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\right)\to H_i\left(Q_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\right)$$
$$\to H_{i-1}\left(C^{\mathrm{gen}}_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}\right)\to\cdots$$

as, by (2), $H_i(Q_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z})=0$ if $i\le 4$ and, by (3), $\mathrm{Im}(H_i(C^{\mathrm{gen}}_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z})\to H_i(C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}))=0$ if $i\le 4$, we deduce that $H_i(C_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z})=0$ for $i\le 4$. Next, we prove these three results, under the hypothesis of the proposition (cf. [19, (3.4), p. 293]):

Lemma 1.14. *The sequence*

$$0\to H_i(G;C^{\mathrm{gen}}_*(n))\to H_i(G;C_*(n))\to H_i(G;Q_*(n))\to 0$$

is exact, for $i\ge 0$, $n\ge 1$.

Proof. For all $i\ge 0$, $C_i(n)\cong C_i^{\mathrm{gen}}\oplus Q_i(n)$. As this decomposition is compatible with the action of G, we get an exact sequence of $\mathbb{Z}G$-modules

$$0\to C^{\mathrm{gen}}_*(n)\to C_*(n)\to Q_*(n)\to 0$$

split as a sequence of $\mathbb{Z}G$-modules (the splitting is not compatible with the differentials). Then we deduce the desired exact sequences. □

Lemma 1.15. *$Q_*(2)\underset{\mathbb{Z}G}{\otimes}\mathbb{Z}$ is 4-acyclic.*

Proof. Let $E^{1,\mathrm{gen}}$, E^1, $E^{1,Q}$ be the spectral sequences associated to the (acyclic) complexes of $\mathbb{Z}G$-modules, $C_*^{\mathrm{gen}}, C_*, Q_*$, converging respectively to $\mathrm{H}_*(G;\mathbb{Z}), \mathrm{H}_*(G;\mathbb{Z})$, $0(=\mathrm{H}_*(G;0))$. As $Q_1(2)=Q_0(2)=0$, we deduce, $E_{s,t}^{1,Q}=0$ with $t\le 1$.

Next, we shall prove that $E_{0,t}^{2,Q}=0$, $0\le t\le 4$. By construction, $E_{s,t}^{\infty,Q}=0$ for all s,t. No (nontrivial) differentials come from $E_{0,3}^{2,Q}$. Thus, $E_{0,3}^{2,Q}=E_{0,3}^{\infty,Q}=0$. Next we show that $E_{1,2}^{2,Q}=0$, and as a consequence we will get $E_{0,4}^{2,Q}=E_{0,4}^{\infty,Q}=0$. But

$$E_{1,2}^{1,Q}=(R^\times\times R^\times)_{2,1}\oplus(R^\times\times R^\times\times R^\times)_{2,2},$$
$$d_{1,3}^{1,Q}((c,a,b)_{3,5})=(a,b,c)_{2,2},$$
$$d_{1,3}^{1,Q}((b,a,a)_{3,2})=(a,b)_{2,1},$$

hence $E_{1,2}^{2,Q}=0$, and this proves the lemma. □

Lemma 1.16. *The map induced in homology by* $C_*^{\mathrm{gen}}(2)\otimes_{\mathbb{Z}G}\mathbb{Z}\to C_*(2)\otimes_{\mathbb{Z}G}\mathbb{Z}$ *is* 0 *in degree lower than* 4.

Proof. We must show that, if z is a cycle in $C_4^{\mathrm{gen}}(2)\otimes_{\mathbb{Z}G}\mathbb{Z}$ then its image in $C_4(2)\otimes_{\mathbb{Z}G}\mathbb{Z}$ is a boundary. Note that $C_3^{\mathrm{gen}}(2)\otimes_{\mathbb{Z}G}\mathbb{Z}\cong\mathbb{Z}$. Then a cycle of $C_4^{\mathrm{gen}}(2)\otimes_{\mathbb{Z}G}\mathbb{Z}$ is a sum of cycles of the form $c-c'$ where c,c' are generic 4-cells. It suffices to prove that cycles of the type $c-c'$ are boundaries in $C_4(2)\otimes_{\mathbb{Z}G}\mathbb{Z}$.

Observe that if $x,y\in R^\times$, then

$$\begin{aligned} d_5&\left(\left([e_1],[e_2],\left[\begin{pmatrix}1\\x\\y\end{pmatrix}\right],[e_3],[e_1+e_2],\left[\begin{pmatrix}1\\x\\0\end{pmatrix}\right]\right)\otimes 1\right)\\ &=([e_1],[e_2],[e_1+e_2+e_3],[e_3],[e_1+e_2])\otimes 1\\ &\quad-\left([e_1],[e_2]\left[\begin{pmatrix}1\\x\\y\end{pmatrix}\right],[e_3],[e_1+e_2]\right)\otimes 1,\end{aligned}$$

because if

$$g=\begin{pmatrix}0 & x^{-1} & 0\\ -1 & 1+x^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad g'=\begin{pmatrix}1 & 0 & -y^{-1}\\ 0 & 1 & -xy^{-1}\\ 0 & 0 & y^{-1}\end{pmatrix}$$

$$g'\left([e_1],[e_2],\left[\begin{pmatrix}1\\x\\y\end{pmatrix}\right],[e_1+e_2],\left[\begin{pmatrix}1\\x\\0\end{pmatrix}\right]\right)$$

$$= \left([e_1], [e_2], [e_3], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right)$$

$$g \left([e_1], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_3], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right)$$

$$= \left([e_2], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_3], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right),$$

and then, if $x, y, x', y' \in R^\times$

$$\left([e_1], [e_2] \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_3], [e_1 + e_2] \right) \otimes 1$$

$$- \left([e_1], [e_2] \left[\begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} \right], [e_3], [e_1 + e_2] \right) \otimes 1$$

is a boundary in $C_4(2) \underset{\mathbb{Z}G}{\otimes} \mathbb{Z}$. In the same way,

$$g \left([e_3], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_1], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right)$$

$$= \left([e_3], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_2], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right)$$

$$g' \left(\left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_1], [e_2], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right)$$

$$= \left([e_3], [e_1], [e_2], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right),$$

and if $x, y, x', y' \in R^\times$, then

$$d_5 \left(\left([e_3], \left[\begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} \right], [e_1], [e_2], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x' \\ 0 \end{pmatrix} \right] \right) \otimes 1 \right.$$

$$\left. - \left([e_3], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \right], [e_1], [e_2], [e_1 + e_2], \left[\begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix} \right] \right) \otimes 1 \right)$$

$$= \left([e_3], \left[\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}\right], [e_1], [e_2], [e_1 + e_2]\right) \otimes 1$$

$$- \left([e_3], \left[\begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix}\right], [e_1], [e_2], [e_1 + e_2]\right) \otimes 1.$$

Let c, c' be generic 4-cells. We can assume that,

$$c = ([e_1], [e_2], v, [e_3], [e_1 + e_2 + e_3]) \otimes 1$$
$$c' = ([e_1], [e_2], v', [e_3], [e_1 + e_2 + e_3]) \otimes 1$$

with

$$v = \left[\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}\right] \quad \text{and} \quad v' = \left[\begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix}\right].$$

By genericity, $\alpha, \beta, 1 - \alpha, 1 - \beta, \alpha - \beta \in R^\times$ (the same for α' and β'). Then

$$\begin{aligned} & d_5(([e_1], [e_2], v, [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \otimes 1) \\ & \quad = ([e_2], v, [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \otimes 1 \\ & \qquad - ([e_1], v, [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \otimes 1 \\ & \qquad + ([e_1], [e_2], [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \otimes 1 \\ & \qquad - ([e_1], [e_2], v, [e_1 + e_2 + e_3], [e_1 + e_2]) \otimes 1 \\ & \qquad + ([e_1], [e_2], v, [e_3], [e_1 + e_2]) \otimes 1 \\ & \qquad - c, \end{aligned}$$

but as

$$\begin{aligned} & g_1([e_2], v, [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \\ & \quad = \left([e_3], \left[\begin{pmatrix} 1 \\ \frac{1}{1-\beta} \\ \frac{1-\alpha}{1-\beta} \end{pmatrix}\right], [e_1], [e_2], [e_1 + e_2]\right) \end{aligned}$$

$$\begin{aligned} & g_2([e_1], v, [e_3], [e_1 + e_2 + e_3], [e_1 + e_2]) \\ & \quad = \left([e_3], \left[\begin{pmatrix} 1 \\ \frac{-\alpha}{\beta-\alpha} \\ \frac{1-\alpha}{\beta-\alpha} \end{pmatrix}\right], [e_1], [e_2], [e_1 + e_2]\right) \end{aligned}$$

$$\begin{aligned} & g_3([e_1], [e_2], v, [e_1 + e_2 + e_3], [e_1 + e_2]) \\ & \quad = \left([e_1], [e_2] \left[\begin{pmatrix} 1 \\ \frac{\alpha-\beta}{1-\beta} \\ \frac{-\beta}{1-\beta} \end{pmatrix}\right], [e_3], [e_1 + e_2]\right) \end{aligned}$$

with

$$g_1=\begin{pmatrix}-1&0&1\\-1&0&0\\-1&1&0\end{pmatrix},\qquad g_2=\begin{pmatrix}0&-1&1\\0&-1&0\\1&-1&0\end{pmatrix},\qquad g_3=\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&-1\end{pmatrix},$$

we see that $c-c'$ is a boundary. □

And this finishes the proof of (1.13). □

As an immediate corollary, we get

Corollary 1.17. $E^{\infty}_{0,i}=E^2_{0,i}=0$ *for* $i\le 4$.

1.3.4. Computations of $E^2_{1,2}$ and $E^2_{1,3}$. Before we go further, we need some more tools. As in [19, p. 295], we can filter $C_*(2)$ in a natural way, by the subcomplex spanned by cells of projective rank lower than one. Therefore, we can use the subcomplex spanned by cells of the type $(v,v',v,v',\dots)$, $v,v'\in\mathbb{P}^2(R)$, modulo the action of G, this corresponds to $([e_1],[e_2],[e_1],[e_2],\dots)$. All these complexes are endowed with the action of G, and induce filtrations on $E^1_{s,\bullet}$.

Set $E^1_{s,\bullet}={}^{(2)}E^1_{s,\bullet}$ with $s\ge 0$. ${}^{(1)}E^1_{s,\bullet}$ is the filtration induced by cells of rank lower than one, and denote the quotient of ${}^{(2)}E^1_{s,\bullet}$ by ${}^{(1)}E^1_{s,\bullet}$ by ${}^{(2/1)}E^1_{s,\bullet}$. We then have,

Proposition 1.18. *For* $t\ge 3$ *and* $s\ge 1$, *the following sequences*

$$0\to E^2_{s,t}\to H_t({}^{(2/1)}E^1_{s,\bullet})\to H_{t-1}({}^{(1)}E^1_{s,\bullet})\to 0 \tag{5}$$

$$0\to E^2_{s,2}\to H_2({}^{(2/1)}E^1_{s,\bullet})\to H_1({}^{(1)}E^1_{s,\bullet})\to E^2_{s,1}\to 0 \tag{6}$$

are exact.

Proof. By construction, there exists an exact sequence of complexes

$$0\to {}^{(1)}E^1_{s,\bullet}\to {}^{(2)}E^1_{s,\bullet}\to {}^{(2/1)}E^1_{s,\bullet}\to 0$$

the homology long exact sequence gives

$$\cdots\to H_t({}^{(1)}E^1_{s,\bullet})\to E^2_{s,t}\to H_t({}^{(2/1)}E^1_{s,\bullet})\to H_{t-1}({}^{(1)}E^1_{s,\bullet})\to\cdots$$

But if $(v_0,\dots,v_n)$ is a cell of rank 1 in C_n, modulo the action of G we can assume that it is of the type $([e_1],[e_2],v'_1,\dots,v'_{n-1})$ with $n\ge 1$ and v'_i are elements of the plane spanned by e_1 and e_2. Thus, if $n\ge 2$,

$$d_{n+1}(([e_3],[e_1],[e_2],v'_1,\dots,v'_n))=([e_1],[e_2],v'_1,\dots,v'_{n-1})+S$$

where S is a sum of cells of rank >1. But the connecting morphism $H_t({}^{(2/1)}E^1_{s,\bullet})\to H_{t-1}({}^{(1)}E^1_{s,\bullet})$ is induced by the differential d_t at the level E^1 and by the projection on the part of rank one. Hence, if $t-1 \geq 2$, the connecting morphism is surjective and we deduce the exact sequence (5). As ${}^{(1)}E^1_{s,i} \cong {}^{(2)}E^1_{s,i}$ for $i=0,1$, we have $H_1({}^{(2/1)}E^1_{s,\bullet})=0=H_0({}^{(2/1)}E^1_{s,\bullet})$. And the end of the long homology exact sequence is just (6). Moreover $H_0({}^{(1)}E^1_{s,\bullet}) \cong E^2_{s,0}$. □

We can now state the following result,

Proposition 1.19.
(1) $E^2_{1,2}=0$.
(2) $E^2_{1,3} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\frac{1}{2}]=0$.

Proof. We show each step.
(1) We apply the exact sequence (6) with $s=1$. We have ${}^{(2/1)}E^1_{1,1}=0$, and ${}^{(2/1)}E^1_{1,2} \cong (R^\times \times R^\times \times R^\times)_{2,0} \cong \operatorname{Ker}(\bar{d}^1_{1,2})$. We then have

$$\bar{d}^1_{1,3}((a^{-1}b,1)_{3,2}) = (ab^{-1},1,1),$$
$$\bar{d}^1_{1,3}((b,c)_{3,6}) = (b,b,c),$$

hence $\bar{d}^1_{1,3}((a^{-1}b,1)_{3,2} + (b,c)_{3,6})=(a,b,c)$ for all $a,b,c \in R^\times$. Thus $\operatorname{Im}(\bar{d}^1_{1,3})= \operatorname{Ker}(\bar{d}^1_{1,2})$, consequently $H_2({}^{(2/1)}E^1_{1,\bullet})=0$, and finally $E^2_{1,2}=0$.
(2) We use (5) with $t=3$ and $s=1$. Show first that $H_2({}^{(1)}E^1_{1,\bullet}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\frac{1}{2}]=0$. The computations gives

$$ {}^{(1)}E^1_{1,2} = (R^\times \times R^\times)_{2,1} \oplus (R^\times \times R^\times \times R^\times)_{2,2},$$
$$\bar{d}^1_{1,2}((a_1,b_1)_{2,1} \oplus (a_2,b_2,c_2)_{2,2}) = (a_1b_2a_2, a_1b_2a_2, b_1c_2^2)_{1,0},$$
$$\operatorname{Ker}(\bar{d}^1_{1,2}) = \{(a^{-1}b^{-1},c^{-2})_{2,1} \oplus (a,b,c)_{2,2},\ a,b,c \in R^\times\}.$$

We then have

$$\bar{d}^1_{1,3}((a,b)_{3,8}) = (a^{-2},b^{-2})_{2,1} + (a,a,b)_{2,2},$$
$$\bar{d}^1_{1,3}((a,b)_{3,10}) = (a^{-1},a^{-1},b^{-1})_{2,2},$$

hence,

$$(a^2,b^2) \equiv 0 \bmod \operatorname{Im}(\bar{d}^1_{1,3}),$$
$$\bar{d}^1_{1,3}((b^{-1}a^{-1},c^{-2})_{3,10} + (b,a,1)_{3,11}) = (ab,ab,c^2)_{2,2} + (ab^{-1},ba^{-1},1)_{2,2}$$
$$= (a^2,b^2,c^2)_{2,2}.$$

This proves that $(\operatorname{Ker}(\bar{d}^1_{1,2}))^2 = \operatorname{Im}(\bar{d}^1_{1,3})$. Show next that $H_3({}^{(2/1)}E^1_{1,\bullet}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\frac{1}{2}]=0$. The main idea is the following:

Suppose that we have abelian groups A,B,C with

$$A \xrightarrow{f} B \xrightarrow{g} C, \qquad g \circ f = 0 \quad \text{and} \quad B = \bigoplus_{i=1}^{N} B_i.$$

Suppose there exists a fixed $j \in \{1,\dots,N\}$, such that for all $x \in \mathrm{Ker}(g)$, there exists $y \in B_j$ and $n \geq 1$, such that $nx \equiv y \bmod \mathrm{Im}(f)$. Then it suffices to show that $g(y)=0$ implies $n'y=0$ for a particular $n' \geq 1$. Begin the proof by the computation of some differentials that we need to get relations in $H_3({}^{(2/1)}E^1_{1,\bullet})$. Consider the following sequence:

$$ {}^{(2/1)}E^1_{1,4} \xrightarrow{\bar{d}^1_{1,4}} {}^{(2/1)}E^1_{1,3} \xrightarrow{\bar{d}^1_{1,3}} {}^{(2/1)}E^1_{1,2} $$

we can see that

$$\begin{aligned} \mathrm{Ker}(\bar{d}^1_{1,3}) \subset (R^\times)_{3,0} &\oplus (R^\times \times R^\times)_{3,1} \oplus (R^\times \times R^\times)_{3,2} \oplus (R^\times \times R^\times)_{3,3} \\ &\oplus (R^\times \times R^\times \times R^\times)_{3,4} \oplus (R^\times \times R^\times \times R^\times)_{3,5} \\ &\oplus (R^\times \times R^\times)_{3,6} \oplus (R^\times \times R^\times \times R^\times)_{3,9}. \end{aligned}$$

As the action of $R^\times$ is trivial in homology, $\bar{d}^1_{1,4} = id$ on the 2-generic component, thus we can ignore $(R^\times)_{3,0}$. Set $B = \mathrm{Im}(\bar{d}^1_{1,4})$. Explicit calculations show that

$$(a,a,a)_{3,4} \equiv 0 \bmod B, \tag{7}$$
$$(b,a)_{3,6} \oplus (a^{-1},b^{-1},b^{-1})_{3,4} \oplus (a,b)_{3,2} \equiv 0 \bmod B, \tag{8}$$
$$(b,a)_{3,2} \oplus (a^{-2},b^{-2},a^{-2})_{3,4} \oplus (a,b)_{3,3} \equiv 0 \bmod B, \tag{9}$$
$$(a,b)_{3,1} \oplus (a^{-1},b^{-1})_{3,3} \oplus (a,b,a)_{3,4} \oplus (a^{-1},a^{-1},b^{-1})_{3,5} \equiv 0 \bmod B, \tag{10}$$
$$(a,b)_{3,1} \oplus (a,b)_{3,6} \equiv 0 \bmod B, \tag{11}$$
$$(a^2,b^2)_{3,3} \oplus (b,a,a)_{3,5} \equiv 0 \bmod B, \tag{12}$$
$$(a^{-1},b^{-1},b^{-1})_{3,5} \oplus (a^2,b^2)_{3,2} \equiv 0 \bmod B, \tag{13}$$
$$(a,b,c)_{3,4} \oplus (b,c,a)_{3,5} \oplus (a,b,c)_{3,9} \equiv 0 \bmod B, \tag{14}$$
$$(b,a)_{3,2} \oplus (a,b)_{3,3} \equiv 0 \bmod B, \tag{15}$$
$$(a^2,b^2)_{3,6} \oplus (a^{-1},a^{-1},b^{-1})_{3,9} \equiv 0 \bmod B, \tag{16}$$
$$(x,x^{-1},1)_{3,9} \equiv 0 \bmod B, \tag{17}$$
$$(1,x,x^{-1})_{3,5} \equiv 0 \bmod B. \tag{18}$$

From (11) and (16) we deduce

$$(a^2,b^2)_{3,1} \oplus (a,a,b)_{3,9} \equiv 0 \bmod B. \tag{19}$$

Putting (15) in (9), and adding the square of (7), we get

$$(1,x^2,1)_{3,4} \equiv 0 \bmod B. \tag{20}$$

By (14), we have

$$(a^2,b^2,b^2)_{3,4} \equiv (b^{-2},b^{-2},a^{-2})_{3,5} \oplus (a^{-2},b^{-2},b^{-2})_{3,9} \bmod B.$$

But with (8), we have

$$(a^2,b^2,b^2)_{3,4} \equiv (b^2,a^2)_{3,6} \oplus (a^2,b^2)_{3,2} \bmod B$$
$$\equiv (b,b,a)_{3,9} \oplus (a,b,b)_{3,5} \bmod B \text{ by (13) and (16).}$$

If we substract these two last relations, we get

$$(b^2a,b^3,a^2b)_{3,5} \oplus (a^2b,b^3,ab^2)_{3,9} \equiv 0 \bmod B,$$

and by using (18) with (17)

$$(1,a^2b^4,ab^2)_{3,9} \oplus (b^2a,b^4a^2,1)_{3,5} \equiv 0 \bmod B,$$

thus

$$(1,x^2,x)_{3,9} \equiv (x^{-1},x^{-2},1)_{3,5} \bmod B. \tag{21}$$

Squaring (10) with $a=1$ and using (20), we get

$$(1,b^2)_{3,1} \oplus (1,b^{-2})_{3,3} \oplus (1,1,b^{-2})_{3,5} \equiv 0 \bmod B,$$

thus with (12) and (19),

$$(1,1,b^{-1})_{3,9} \oplus ((b,1,1)_{3,5} + (1,1,b^{-2})_{3,5}) \equiv 0 \bmod B;$$

hence

$$(1,1,b)_{3,9} \equiv (b,1,b^{-2})_{3,5} \bmod B. \tag{22}$$

And by combining the previous results, we get

$$(a^2,b^2,c)_{3,9} \equiv (1,a^2b^2,c)_{3,9} \bmod B \quad \text{by (17)}$$
$$\equiv (1,a^2b^2,ab)_{3,9} + (1,1,a^{-1}b^{-1}c)_{3,9} \bmod B$$
$$\equiv (ca^{-2}b^{-2},a^{-2}b^{-2},a^2b^2c^{-2})_{3,5} \bmod B \quad \text{by (21) and (22).}$$

Call a component $(3,i)$ an element of the type $(a,b,c)_{3,i}$, $i=4,5,9$, or of the type $(a,b)_{3,i}$, $i=1,2,3,6$.

The previous result shows that, if x is a component $(3,9)$ then x^2 is homologous to a component $(3,5)$. As by (14), a component $(3,4)$ is homologous to a sum of components $(3,5)$ and $(3,9)$, we deduce that if x is a component $(3,4)$, x^2 is homologous to a component $(3,5)$. If x is a component $(3,1)$ or $(3,6)$ then by (16) or (19), x^2 is homologous to a component $(3,9)$, and then x^4 is homologous to a component $(3,5)$. Finally, if x is a component $(3,2)$ or $(3,3)$, by (12) or (13), we see that x^2 is a component $(3,5)$. Consequently, if z is a cycle of ${}^{(2/1)}E^1_{1,3}$, z^4 is homologous to a component $(3,5)$ which is necessarily a cycle. But if $(a,b,c)_{3,5}$ is a cycle, $\bar{d}^1_{1,3}((a,b,c)_{3,5})=(a^{-2},b^{-1}c^{-1},b^{-1}c^{-1})=(1,1,1)$, thus $a^2=1$ and $b^{-1}=c$. Hence $(a,b,c)_{3,5}=(a,b,b^{-1})_{3,5}$. But, by (18), $(1,b,b^{-1})_{3,5} \equiv 0 \bmod B$. And as $(a,1,1)^2_{3,5}=0$, we conclude that $z^8 \equiv 0 \bmod B$, and this finishes the proof. □

1.3.5. Computation of $E^2_{2,2}$. The last information we need to prove our first main result is the following,

Proposition 1.20. $E^2_{2,2} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\frac{1}{3}] = 0.$

Proof. By (6) with $s=2$, the term $E^2_{2,2}$ is the kernel of the connecting morphism $H_2({}^{(2/1)}E^1_{2,\bullet}) \xrightarrow{\partial} H_1({}^{(1)}E^1_{2,\bullet})$. Thus it is, in particular, a subgroup of $H_2({}^{(2/1)}E^1_{2,\bullet})$. We have the following sequence of morphism,

$$ {}^{(2/1)}E^1_{2,3} \xrightarrow{\bar{d}^1_{2,3}} {}^{(2/1)}E^1_{2,2} \to 0 $$

and

$$ {}^{(2/1)}E^1_{2,2} \cong \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times). $$

The generic components of ${}^{(2/1)}E^1_{2,3}$ are given by $s_{3,i}$ $i=1,\dots,6$ and $s_{3,9}$. $s_{3,0}$ giving no contribution. We have,

$$ \begin{aligned} \bar{d}^1_{2,3}((a,a,b)\wedge(a',a',b')_{3,1}) &= -(a,a,b)\wedge(a',a',b') \\ \bar{d}^1_{2,3}((a,b,b)\wedge(a',b',b')_{3,2}) &= -(a,b,b)\wedge(a',b',b') \\ \bar{d}^1_{2,3}((a,b,a)\wedge(a',b',a')_{3,3}) &= (b,a,a)\wedge(b',a',a') \\ \bar{d}^1_{2,3}((a,b,c)\wedge(a',b',c')_{3,4}) &= (b,c,a)\wedge(b',c',a') - (a,b,c)\wedge(a',b',c') \\ \bar{d}^1_{2,3}((a,b,c)\wedge(a',b',c')_{3,5}) &= -(a,c,b)\wedge(a',c',b') - (a,b,c)\wedge(a',b',c') \\ \bar{d}^1_{2,3}((a,a,b)\wedge(a',a',b')_{3,6}) &= (a,a,b)\wedge(a',a',b') \\ \bar{d}^1_{2,3}((a,b,c)\wedge(a',b',c')_{3,9}) &= (b,a,c)\wedge(b',a',c') + (a,b,c)\wedge(a',b',c'). \end{aligned} $$

As ${}^{(2/1)}E^1_{2,2} \cong \bigwedge^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)$, we can identify the group $H_2({}^{(2/1)}E^1_{2,\bullet})$ to

$$ \frac{\bigwedge^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)}{N_1}, $$

where N_1 is the subgroup of $\bigwedge^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)$ spanned by the following elements:

$$ (a,a,b)\wedge(a',a',b') \tag{23} $$

$$ (a,b,b)\wedge(a',b',b') \tag{24} $$

$$ (a,b,a)\wedge(a',b',a') \tag{25} $$

$$ (b,c,a)\wedge(b',c',a') - (a,b,c)\wedge(a',b',c') \tag{26} $$

$$ (a,b,c)\wedge(a',b',c') + (a,c,b)\wedge(a',c',b') \tag{27} $$

$$ (b,a,c)\wedge(b',a',c') + (a,b,c)\wedge(a',b',c') \tag{28} $$

where a, b, c, a', b', c' are elements of $R^\times$. We add (25), which is a consequence of (24) and (26), because it is useful for the sequel. In the same way, $H_1({}^{(1)}E^1_{2,\bullet})$ is given by the following complex:

$$
{}^{(1)}E^1_{2,2} \xrightarrow{\bar{d}^1_{2,2}} {}^{(1)}E^1_{2,1} \xrightarrow{\bar{d}^1_{2,1}} {}^{(1)}E^1_{2,0},
$$

as

$$
\begin{aligned}
{}^{(1)}E^1_{2,2} &\cong \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)_{2,2} \oplus \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times)_{2,1} \\
{}^{(1)}E^1_{2,1} &\cong \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times),
\end{aligned}
$$

we can write the differentials as follows:

$$
\begin{aligned}
&\bar{d}^1_{2,2}((a,b,c) \wedge (a',b',c')_{2,2}) = (b,a,c) \wedge (b',a',c') + (a,b,c) \wedge (a',b',c') \\
&\bar{d}^1_{2,2}((a,a,b) \wedge (a',a',b')_{2,1}) = (a,a,b) \wedge (a',a',b').
\end{aligned}
$$

Then, $H_1({}^{(1)}E^1_{2,\bullet})$ is a subgroup of

$$
\frac{\bigwedge^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)}{N_2},
$$

where N_2 is spanned by the following elements:

$$
\begin{aligned}
&(b,a,c) \wedge (b',a',c') + (a,b,c) \wedge (a',b',c') \\
&(a,a,b) \wedge (a',a',b').
\end{aligned}
$$

We note the important fact, that N_2 is a subgroup of N_1. Then, we can see a relation in

$$
\frac{\bigwedge^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times)}{N_2}
$$

as a relation in $H_2({}^{(2/1)}E^1_{2,\bullet})$. We want to show now, that $E^2_{2,2}$ is killed by 3. We have the following isomorphism:

$$
\begin{aligned}
\xi : \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times \times R^\times) &\to \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times \times R^\times) \oplus \left[(R^\times \times R^\times) \otimes R^\times\right] \oplus \bigwedge\nolimits^2_{\mathbb{Z}} (R^\times) \\
(a,b,c) \wedge (a',b',c') &\mapsto (a,b) \wedge (a',b') \oplus [(a,b) \otimes c' - (a',b') \otimes c] \oplus c \wedge c'.
\end{aligned}
$$

Set $\hat{N}_1 = \xi(N_1)$. Now, we can write the relations (23)–(25) as:

$$
(a,a) \wedge (a',a') \oplus [(a,a) \otimes b' - (a',a') \otimes b] \oplus b \wedge b' \equiv 0 \bmod \hat{N}_1 \tag{29}
$$

$$
(a,b) \wedge (a',b') \oplus [(a,b) \otimes b' - (a',b') \otimes b] \oplus b \wedge b' \equiv 0 \bmod \hat{N}_1 \tag{30}
$$

$$
(a,b) \wedge (a',b') \oplus [(a,b) \otimes a' - (a',b') \otimes a] \oplus a \wedge a' \equiv 0 \bmod \hat{N}_1. \tag{31}
$$

Applying (29) with $a=a'=1$, we get, for all $b,b'\in R^\times$, $b\wedge b'\equiv 0 \bmod \hat{N}_1$. Then we can forget $\bigwedge^2_{\mathbb{Z}}(R^\times)$. Applying (30) with $b=b'=1$, we get

$$(a,1)\wedge(a',1)\equiv 0 \bmod \hat{N}_1 \quad \text{for all } a,a'\in R^\times. \tag{32}$$

Next applying (31) with $a=a'=1$, we get

$$(1,b)\wedge(1,b')\equiv 0 \bmod \hat{N}_1 \quad \text{for all } b,b'\in R^\times. \tag{33}$$

Then we deduce that the non trivial elements of $\bigwedge^2_{\mathbb{Z}}(R^\times \times R^\times)$ are the $(1,x)\wedge(y,1)$, since by (28) (with $c=c'=1$), we have $(a,b)\wedge(a',b')\equiv -\ (b,a)\wedge(b',a') \bmod \hat{N}_1$. Moreover, by using (30) with $a=b'=1$, we get

$$(a',1)\otimes b\equiv(1,b)\wedge(a',1) \bmod \hat{N}_1, \tag{34}$$

and as, by (29) with $a'=1$, we get $(a,1)\otimes b'\equiv -(1,a)\otimes b' \bmod \hat{N}_1$. We see that every element of $H_2({}^{(2/1)}E^1_{2,\bullet})$ is a sum of elements of the type $(1,x)\wedge(y,1)$.

Let z be an arbitrary element of $E^2_{2,2}$. Then by our previous assertions, we can assume that, modulo $\hat{N}_1$, $z=\sum n_i(1,a_i)\wedge(b_i,1)$, $n_i\in\mathbb{Z}$, $a_i,b_i\in R^\times$.

Look at $z=\sum n_i(1,a_i,1)\wedge(b_i,1,1)$ in $\bigwedge^2_{\mathbb{Z}}(R^\times\times R^\times\times R^\times)$. The computation of $d^1_{p,2}$ gives

$$\begin{aligned} x &= d^1_{2,2}((a,b,c)\wedge(a',b',c')) \\ &= (b,c,a)\wedge(b',c',a') - (a,c,b)\wedge(a',c',b') + (a,b,c)\wedge(a',b',c'). \end{aligned}$$

As a special case, if $a=c=b'=c'=1$, then

$$\xi(x)=(1,b)\wedge(a',1)\oplus[(b,1)\otimes a' + (a',1)\otimes b].$$

Moreover, using (31) with $b=b'=1$ and (32), we get the new relation

$$(a,1)\otimes a'\equiv(a',1)\otimes a \bmod \hat{N}_1, \tag{35}$$

then we deduce that

$$\begin{aligned} \partial(z) &= \sum n_i((1,a_i)\wedge(b_i,1)\oplus[(a_i,1)\otimes b_i + (b_i,1)\otimes a_i]) \\ &= 0 \end{aligned}$$

and by (35), implies that

$$3\left(\sum n_i(1,a_i)\wedge(b_i,1)\right)\equiv 0 \bmod \hat{N}_1,$$

in other words, $3z=0$, and this ends the proof. □

Summarizing, we have

Corollary 1.21. $E^{\infty}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]=E^{2}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]$.

Proof. First, we notice that $d^{r}_{3,0}=0$ for all $r\geq 1$. By (1.20) we have $d^{2}_{2,2}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{3}]=0$, then $E^{3}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{3}]=E^{2}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{3}]$. By (1.19(2)) we have $d^{3}_{1,3}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{2}]=0$, then $E^{4}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]=E^{2}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]$. And finally by (1.17) $d^{4}_{0,4}=0$, hence $E^{\infty}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]=E^{4}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]=E^{2}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{6}]$. □

Now, we can state the central result

Theorem 1.22. *Let R be a commutative* H1*-ring. Then the morphism* $\mathrm{H}_3(GL_2(R);\mathbb{Q})\to\mathrm{H}_3(GL_3(R);\mathbb{Q})$, *induced by* $GL_2(R)\to GL_3(R)$, $g\mapsto\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}$, *is injective.*

Proof. Denote by $\Gamma_{\bullet}$ a $\mathbb{Z}G$-resolution of $\mathbb{Z}$, where, as previously, $G=GL_3(R)$. Recall that $B_{0,0}=Stab_G([e_1])$, and identify $GL_2(R)$ with its image in $GL_3(R)$, via the stabilization morphism. Then, $GL_2(R)\leq B_{0,0}\leq G$, and we can see $\Gamma_{\bullet}$ as a $\mathbb{Z}B_{0,0}$ (resp. $\mathbb{Z}GL_2(R)$) - resolution of $\mathbb{Z}$.

We have a commutative diagram,

$$\begin{array}{ccc}
\Gamma_{\bullet}\otimes_{\mathbb{Z}GL_2(R)}\mathbb{Z} & \longrightarrow & \Gamma_{\bullet}\otimes_{\mathbb{Z}G}(\mathbb{Z}G\otimes_{\mathbb{Z}GL_2(R)}\mathbb{Z})\\
\downarrow & & \downarrow\\
\Gamma_{\bullet}\otimes_{\mathbb{Z}B_{0,0}}\mathbb{Z} & \longrightarrow & \Gamma_{\bullet}\otimes_{\mathbb{Z}G}(\mathbb{Z}G\otimes_{\mathbb{Z}B_{0,0}}\mathbb{Z})
\end{array}$$

the different maps are given by the stabilization morphism and the induction [7, Section III.5, pp. 67–69]. This give us the commutativity of

$$\begin{array}{ccc}
\mathrm{H}_3(GL_3(R);\mathbb{Q}) & \xrightarrow{(1)} & E^{\infty}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Q}\\
{\scriptstyle (2)}\uparrow & & \uparrow\cong\\
\mathrm{H}_3(GL_2(R);\mathbb{Q}) & \xrightarrow[(3)]{} & E^{2}_{3,0}\otimes_{\mathbb{Z}}\mathbb{Q}
\end{array}$$

where (1) is the arrow coming from the abutment, induced by the filtration of the bicomplex $\Gamma_{\bullet}\otimes_{\mathbb{Z}G}C_{\bullet}$, (2) is induced by the stabilization morphism, and (3) is the map of (1.8).

Summarizing, the injectivity of (3) implies those of (2). □

We deduce from this:

Corollary 1.23. *If R is a commutative* H1*-ring, then the morphism* $\overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q}) \to \overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q})$, *induced by $GL_2(R) \rightarrowtail GL_3(R)$ at the LHS level, is injective, where* $\overline{\mathrm{H}}_3(SL_p(R);\mathbb{Q})$ *is a useful notation for* $\mathrm{H}_0(R^\times;\mathrm{H}_3(SL_p(R);\mathbb{Q}))$, *with* $p=2,3$.

Proof. It is an immediate corollary of (A.8). We have the following commutative diagram:

$$\begin{array}{ccc}
\overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q}) & \longrightarrow & \mathrm{H}_3(GL_2(R);\mathbb{Q}) \\
\downarrow & & \downarrow j \\
\overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q}) & \longrightarrow & \mathrm{H}_3(GL_3(R);\mathbb{Q})
\end{array}$$

the horizontal arrows are injective by (A.8) and j is injective by (1.22). □

Remark 1.24. (1) In (1.8) we tensor by $\mathbb{Q}$ for simplicity, because in this case for an abelian group A, $\mathrm{H}_*(A;\mathbb{Q}) \cong \bigwedge^*_{\mathbb{Q}}(A)$. But we expect that the results are also true if we use $\mathbb{Z}[\frac{1}{2}]$ as coefficient.

(2) Notice that the torsions that we found is different from those announced by Sah in [19, (3.19), p. 303].

2. The relationship with the indecomposable K_3 of rings and the homology of SL_2

By [13, Section 4, pp. 57–58], we have, for all $n \geq 0$, a morphism constructed in the following way:

$$\varphi_n : \mathrm{K}_n(R) \xrightarrow{\text{Hurewicz}} \mathrm{H}_n(GL(R);\mathbb{Z}) \xleftarrow[\cong]{\text{stabilization}} \mathrm{H}_n(GL_n(R);\mathbb{Z}) \xrightarrow{\text{obstruction}} \mathrm{K}_n^{\mathrm{M}}(R).$$

The "obstruction" morphism is given by the composition of ε_n [13, (3.3), p. 50, (3.3.2) p. 51] and η_n [13, (3.3.6) p. 55]. Since R is H1, $\mathrm{K}_1^{\mathrm{M}}(R) \cong \mathrm{K}_1(R)$. The K-theory product [18, (5.3.1), p. 280] gives the morphism:

$$\psi_n : \mathrm{K}_n^{\mathrm{M}}(R) \to \mathrm{K}_n(R), \quad n \geq 1.$$

Then, by [13, (4.1.1), p. 58], we know that

$$\varphi_n \circ \psi_n : \mathrm{K}_n^{\mathrm{M}}(R) \to \mathrm{K}_n(R) \to \mathrm{K}_n^{\mathrm{M}}(R)$$

is the multiplication by $(-1)^{n-1}(n-1)!$. As in the case of fields, we define $\mathrm{K}_3(R)^{\mathrm{ind}}_{\mathbb{Q}}$ as the quotient $\mathrm{K}_3(R)_{\mathbb{Q}}/\mathrm{K}_3^{\mathrm{M}}(R)_{\mathbb{Q}}$.

Remark 2.1. Rationally, the composition

$$p : \mathrm{Ker}\,(\varphi_3) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \overset{\text{can}}{\hookrightarrow} K_3(R)_{\mathbb{Q}} \xrightarrow{\text{projection}} \mathrm{K}_3(R)^{\mathrm{ind}}_{\mathbb{Q}},$$

where "can" is the canonical embedding, is an isomorphism. Thus we can identify $\mathrm{K}_3(R)^{\mathrm{ind}}_{\mathbb{Q}}$ and $\mathrm{Ker}\,(\varphi_3) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, and also with $\mathrm{Coker}\,(\tilde{\psi}_3) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

We now prove that

Theorem 2.2. *Let R be a (commutative)* H1*-ring. We have an isomorphism*:

$$\mathrm{K}_3(R)^{\mathrm{ind}}_{\mathbb{Q}} \cong \overline{\mathrm{H}}_3(SL_2(R); \mathbb{Q}).$$

Before proving (2.2), we establish the analog for rings of [12, 5.15(ii), p. 123].

Lemma 2.3. *If R is an* H1*-ring, then*

$$\frac{\overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q})}{\overline{\mathrm{H}}_3(SL_2(R); \mathbb{Q})} \cong \mathrm{K}^{\mathrm{M}}_3(R)_{\mathbb{Q}}.$$

Proof. By (4.8(1)) we have $\overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q}) \hookrightarrow \mathrm{H}_3(GL_3(R); \mathbb{Q})$, moreover by Guin [13, Théorème 2, pp. 44–45] and by (1.22), we have an exact sequence

$$0 \to \mathrm{H}_3(GL_2(R); \mathbb{Q}) \to \mathrm{H}_3(GL_3(R); \mathbb{Q}) \xrightarrow{\pi} \mathrm{K}^{\mathrm{M}}_3(R)_{\mathbb{Q}} \to 0$$

where π is the "obstruction" morphism. As we have the isomorphisms:

$\overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q}) \cong \mathrm{H}_3(SL(R); \mathbb{Q})$	(4.8(2))
$SL(R) = E(R)$	[9, (1.1.11), p. 9]
$\mathrm{H}_3(E(R); \mathbb{Q}) \cong \mathrm{K}_3(R)_{\mathbb{Q}}$	[19, 2.5 (a), p. 282; 3, (1.6a), p. 5, Section 1.9, p. 6][1]

we deduce that the morphism $\hat{\pi} : \overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q}) \to \mathrm{K}^{\mathrm{M}}_3(R)_{\mathbb{Q}}$, restriction of π to $\overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q})$, is onto. Hence, we have the exact sequence

$$0 \to \mathrm{Ker}\,(\hat{\pi}) \to \overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q}) \to \mathrm{K}^{\mathrm{M}}_3(R)_{\mathbb{Q}} \to 0$$

but $\mathrm{Ker}\,(\hat{\pi}) = \overline{\mathrm{H}}_3(SL_3(R); \mathbb{Q}) \cap \mathrm{H}_3(GL_2(R); \mathbb{Q})$ (using (1.22)), and finally by (A.8(3)) and (1.23) we get $\mathrm{Ker}\,(\hat{\pi}) = \overline{\mathrm{H}}_3(SL_2(R); \mathbb{Q})$. □

Proof of Theorem 2.2. By (2.1), we have an exact sequence

$$0 \to \mathrm{K}^{\mathrm{M}}_3(R)_{\mathbb{Q}} \xrightarrow{\tilde{\psi}_3} \mathrm{K}_3(R)_{\mathbb{Q}} \xrightarrow{\mathrm{pr}} \mathrm{K}_3(R)^{\mathrm{ind}}_{\mathbb{Q}} \to 0$$

[1] See also [22, (5.1) and (5.2), p. 231].

and by using the arguments of (2.3), we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q}) & \longrightarrow & \overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q}) & \xrightarrow{\tilde{\pi}} & \mathrm{K}_3^{\mathrm{M}}(R)_{\mathbb{Q}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & \| & & \\
0 & \longrightarrow & \mathrm{K}_3(R)_{\mathbb{Q}}^{\mathrm{ind}} & \xrightarrow{\mathrm{can}\circ p^{-1}} & \mathrm{K}_3(R)_{\mathbb{Q}} & \xrightarrow{\varphi_3} & \mathrm{K}_3^{\mathrm{M}}(R)_{\mathbb{Q}} & \longrightarrow & 0
\end{array}$$

where g is the inverse of the map $\mathrm{K}_3(R)_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{H}_3(SL(R);\mathbb{Q}) \cong \overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q})$, can and p are the maps defined in (2.1), and f is induced by the commutativity of the right part of the diagram. We conclude by the "five lemma". □

Let

$$F_j^{\mathrm{rank}}\mathrm{K}_n(R)_{\mathbb{Q}} = \mathrm{Im}(\mathrm{H}_n(GL_j(R);\mathbb{Q}) \to \mathrm{H}_n(GL(R);\mathbb{Q})) \cap Prim\,\mathrm{H}_n(GL(R);\mathbb{Q})$$

for $n \geq 0$ and $1 \leq j \leq n$.

Corollary 2.4. *We have*

$$F_j^{\mathrm{rank}}\mathrm{K}_n(R)_{\mathbb{Q}} \oplus F_{\gamma}^{j+1}\mathrm{K}_n(R)_{\mathbb{Q}} = \mathrm{K}_n(R)_{\mathbb{Q}}, \textit{ for } 1 \leq n \leq 3 \textit{ and } 1 \leq j \leq n.$$

Here $F_{\gamma}^{\bullet}$ denotes the γ-filtration of the K-theory [20, (1.5) p. 493]. Before the proof of (2.4), we need the analog of a result of [20, Théorème 2 p. 502]

Proposition 2.5. *If R is an* H1*-ring, then for $i \geq 1$*

(1) $\mathrm{K}_i^{\mathrm{M}}(R) \xrightarrow{\cong} F_{\gamma}^{i}\mathrm{K}_i(R)$ (modulo $\mathcal{S}_i$) where $\mathcal{S}_i$ is the Serre class of abelian groups A, such that there exist an integer m (depending of A) which satisfies $ma = 0$ if $a \in A$ and if p is any prime number dividing m we then have $p = 2$ or $p < i$.

(2) $K_i^{(i)}(R)_{\mathbb{Q}} \cong gr_{\gamma}^{i}\mathrm{K}_i(R)_{\mathbb{Q}} \cong \mathrm{K}_i^{\mathrm{M}}(R)_{\mathbb{Q}}$, where $K_i^{(i)}(R)_{\mathbb{Q}}$ denotes the component of weight i for the Adams operation Ψ^i.

(3) $K_3(R)_{\mathbb{Q}}^{\mathrm{ind}} \cong K_3^{(2)}(R)_{\mathbb{Q}}$, where $K_3^{(2)}(R)_{\mathbb{Q}}$ denotes the component of weight 2 for the Adams operation Ψ^3.

Proof. (1) In fact we can extend the proof of Soulé [20, 3.3, pp. 506–507] word for word, except[2] for the proof of Proposition 3 [20, p. 507] which is not quite complete (see below). For the other parts of the proof we use the results of Guin [13, Théorème 1, pp. 33–34, Théorème 2, pp. 44–45, (4.1.1), p. 58] and the fact that $BSL(R)^+ \times BR^\times \simeq BGL(R)^+$. For the convenience of the reader we recall the last homotopy equivalence. We have $[GL(R), GL(R)] = E(R)$, thus $GL(R)$ is quasiperfect [15, (1.1.6) p. 317]; moreover we have a structure of direct sum [15, (1.2.5) p. 320, (1.3) p. 323]. As the ring is of stable rank one, $SL(R) = E(R)$, and then the morphism

[2] D. Arlettaz pointed out this erratum to me.

$GL(R) \to GL(R)^{ab} \cong R^\times$ has a section s, which is a group morphism. Thus we deduce the composition

$$SL(R) \times R^\times \overset{\mathrm{can}\times s}{\longrightarrow} GL(R) \times GL(R) \overset{\oplus}{\to} GL(R)$$

where can is the canonical embedding, and at the level of the "+"-construction, we get

$$BSL(R)^+ \times BR^\times \to BGL(R)^+.$$

And now we can apply [21, (5.3) p. 232], to get the desired homotopy equivalence. For the end of the proof we proceed as follow: denote by $h_i : \mathrm{K}_i(R) \to \mathrm{H}_i(SL(R);\mathbb{Z})$ the Hurewicz morphism. As for $i=1$, the assertion (1) is trivial, we must show that for $i \geq 2$, $\mathrm{Ker}(h_i)$ is in $\mathscr{S}_i$. We apply [3, (1.6)(a), p. 5]. Then $\mathrm{Ker}(h_i)$ is killed by R_{i-1} (cf. [3, (1.3), p. 4] for the definition of the integers R_l). Suppose that p divides R_{i-1}, then by [3, (1.3), p. 4], we have $p \leq (\frac{i-1}{2}) + 1$ and as $i \geq 2$, we get $p < i$.

(2) It's because $F_\gamma^{i+1}\mathrm{K}_i(R) = 0$ [20, Théorème 1(ii), p. 494] and rationally the decompositions are the same.

(3) Consequence of [20, Corollaire 1, p. 498] and of (2).

Proof of Corollary 2.4. We just prove the case $n=3, j=2$. Denote by h the isomorphism between $\mathrm{K}_3^{\mathrm{M}}(R)_{\mathbb{Q}}$ and $F_\gamma^3\mathrm{K}_3(R)_{\mathbb{Q}}$ (consequence of (2.5(2))). Recall that we have an isomorphism $g : \overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q}) \to \mathrm{K}_3(R)_{\mathbb{Q}}$. Let $p_1 = h \circ \hat{\pi} \circ g^{-1}$, and set $N = \mathrm{Ker}(p_1)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q}) & \longrightarrow & \overline{\mathrm{H}}_3(SL_3(R);\mathbb{Q}) & \overset{\hat{\pi}}{\longrightarrow} & \mathrm{K}_3^{\mathrm{M}}(R)_{\mathbb{Q}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & {\scriptstyle h}\downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & N & \longrightarrow & \mathrm{K}_3(R)_{\mathbb{Q}} & \overset{p_1}{\longrightarrow} & F_\gamma^3\mathrm{K}_3(R)_{\mathbb{Q}} & \longrightarrow & 0
\end{array}$$

where ι is induced by the commutativity of the right part of the diagram. Thus $N = \overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q})$. By (1.23), $F_2^{\mathrm{rank}}\,\mathrm{K}_3(R)_{\mathbb{Q}} = \mathrm{H}_3(GL_2(R);\mathbb{Q}) \cap \mathrm{K}_3(R)_{\mathbb{Q}}$.

But $\overline{\mathrm{H}}_3(SL_2(R);\mathbb{Q}) = \mathrm{H}_3(GL_2(R);\mathbb{Q}) \cap \mathrm{K}_3(R)_{\mathbb{Q}}$, which finishes the proof. □

3. Further comments

In [5, (7.6) p. 699], Borel and Yang proved that in the case of a number field F (or $F = \bar{\mathbb{Q}}$), the morphism $\rho_{i,n} : \mathrm{H}_i(GL_n(F);\mathbb{Q}) \to \mathrm{H}_i(GL_{n+1}(F);\mathbb{Q})$, is always injective for $i \geq 1$ and $n \geq 1$. I don't know if the result remains true if we replace F by an infinite field [19, Problem 4.13, p. 307]. As there is a dictionary between K-theory and cyclic homology (linear groups and Lie algebra, *see* [16] for instance), we can ask for the analog of (2.2) in the case of Lie algebra,[3] but this was already done by Cathelineau in [8]. Moreover, in this paper he computes exactly the groups involved.

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Appendix

For the notations and definitions in this section (and this paper) see [7, Ch. III, paragraphs 4–10].

A.1. Morphism induced in homology via "Eckmann–Shapiro lemma"

The context is as follows: let G_1 and G_2 be two groups, with a morphism $\rho : G_1 \to G_2$, and X_1 (resp. X_2) be a G_1 (resp. G_2)-set. Suppose there exist $x_1 \in X_1$ and $x_2 \in X_2$, such that $X_i = G_i.x_i$, with $i = 1,2$. Denote by H_i the stabilizer of x_i in G_i. Then we can characterize every map $\varphi : \mathbb{Z}[X_1] \to \mathbb{Z}[X_2]$, such that $\varphi(g.x) = \rho(g).\varphi(x)$, with $g \in G_1, x \in X_1$, by its value on x_1. Let $\varphi(x_1) = \sum_{g \in E_2} n_g g x_2$, $n_g \in \mathbb{Z}$, where n_g depends of φ and E_2 denotes a set of representative for G/H_2.

Thus the pair (φ, ρ) induces a map $\bar{\varphi} : \mathrm{H}_*(G_1; \mathbb{Z}[X_1]) \to \mathrm{H}_*(G_2; \mathbb{Z}[X_2])$, which depends of ρ. But by "Eckmann–Shapiro lemma"[4] (see [7, (6.2) p. 73]), we can identify this map to a map $\bar{\varphi} : \mathrm{H}_*(H_1; \mathbb{Z}) \to \mathrm{H}_*(H_2; \mathbb{Z})$. The problem is to give a description of this map. We now give a generalisation of a result of Hutchinson [14, Lemma 3, p. 183]. First denote by $\tilde{H}_1$ the image of H_1 by ρ.

Proposition A.1. (1) *Let $E = \tilde{H}_1 \backslash G/H_2$. Then n_g depends only on the class of g in E.*
(2) *If $n_g \neq 0$ then $[\tilde{H}_1 : \tilde{H}_1 \cap gH_2g^{-1}] < \infty$.*
(3) *The map $\bar{\varphi}$ is given by the formula*

$$\bar{\varphi}(z) = \sum_{g \in E} n_g \operatorname{cor}^{H_2}_{g^{-1}\tilde{H}_1 g \cap H_2} \operatorname{res}^{g^{-1}\tilde{H}_1 g}_{g^{-1}\tilde{H}_1 g \cap H_2} g^{-1} \bar{\rho}(z)$$

where $\bar{\rho}$ is the map induced in homology by ρ.

[3] It is D. Guin who pointed out this question to me.

[4] Historically the wellknown "Shapiro lemma" was proved by B. Eckmann.

(4) *If for every* $g \in E$, *such that* $n_g \neq 0$, $\tilde{H}_1 \leq gH_2g^{-1}$ *then*

$$\bar{\varphi}(z) = \sum_{g \in E_2} n_g \operatorname{cor}^{H_2}_{g^{-1}\tilde{H}_1 g} g^{-1} \bar{\rho}(z).$$

Then we can get the generalisation to the non-transitive case,

Proposition A.2. *Let* I *and* J *be subsets of* $\mathbb{N} - \{0\}$, $X_{1,i}$, $X_{2,j}$ *G-sets with* $(i,j) \in I \times J$, $X_{1,i} = G.x_{1,i}$, $X_{2,j} = G.x_{2,j}$, $x_{1,i} \in X_{1,i}$, $x_{2,j} \in X_{2,j}$. *Denote by* $H_{1,i}$ *(resp.* $H_{2,j}$*) the stabilizer of* $x_{1,i}$ *(resp.* $x_{2,j}$*) in* G.

Let

$$\psi : \bigoplus_{i \in I} \mathbb{Z}[X_{1,i}] \to \bigoplus_{j \in J} \mathbb{Z}[X_{2,j}]$$

Then

$$\bar{\psi} : \bigoplus_{i \in I} \mathrm{H}_*(H_{1,i}; \mathbb{Z}) \to \bigoplus_{j \in J} \mathrm{H}_*(H_{2,j}; \mathbb{Z})$$

is given by the formula

$$\bar{\psi}(z) = \sum_{i \in I} \sum_{j \in J} \operatorname{can}_j \left(\sum_{g \in E_{i,j}} n_g \operatorname{cor}^{H_{2,j}}_{g^{-1}H_{1,i}g \cap H_{2,j}} \operatorname{res}^{g^{-1}H_{1,i}g}_{g^{-1}H_{1,i}g \cap H_{2,j}} g^{-1} \operatorname{pr}_i(z) \right)$$

where $E_{i,j}$ *is a set of representatives of* $H_{1,i} \backslash G / H_{2,j}$, *the* n_g *are determined by* $\psi_{i,j} = \operatorname{pr}_j \circ \psi \circ \operatorname{can}_i$ (can *and* pr, *may be with indices, denote the canonical mophisms associated to direct sums and products of* $\mathbb{Z}$*-modules*).

Moreover, if for all g, *with* $n_g \neq 0$, *we have* $H_{1,i} \leq gH_{2,j}g^{-1}$ *for all* $(i,j) \in I \times J$, *then*:

$$\bar{\psi}(z) = \sum_{i \in I} \sum_{j \in J} \operatorname{can}_j \left(\sum_{g \in E_j} n_g \operatorname{cor}^{H_{2,j}}_{g^{-1}H_{1,i}g} g^{-1} \operatorname{pr}_i(z) \right)$$

where E_j *is a set of representatives of* $G/H_{2,j}$. *(Notice that* n_g *depends of* ψ, i *and* j*).*

Proof. By (A.1), we know that if $z_i \in \mathrm{H}_*(H_{1,i}; \mathbb{Z})$, then

$$\bar{\psi}_{i,j} = \sum_{g \in E_{i,j}} n_g \operatorname{cor}^{g^{-1}H_{1,i}g}_{g^{-1}H_{1,i}g \cap H_{2,j}} \operatorname{res}^{g^{-1}H_{1,i}g}_{g^{-1}H_{1,i}g \cap H_{2,j}} g^{-1} z_i.$$

By [6, Section II.12, paragraphe 6], if

$$z \in \bigoplus_{i \in I} \mathrm{H}_*(H_{1,i}; \mathbb{Z}),$$

we have $z = \sum_{i \in I} \operatorname{can}_i(\operatorname{pr}_i(z))$ and

$$\bar{\psi}(z) = \sum_{j \in J} \operatorname{can}_j(\operatorname{pr}_j(\bar{\psi}(z)),$$

where $\bar{\psi} = \sum_{i\in I}\sum_{j\in J} \mathrm{can}_j \circ \mathrm{pr}_j \circ \bar{\psi} \circ \mathrm{can}_i \circ \mathrm{pr}_i$. Thus

$$\bar{\psi}(z) = \sum_{i\in I}\sum_{j\in J} \mathrm{can}_j \left(\sum_{g\in E_{i,j}} n_g \,\mathrm{cor}^{H_{2,j}}_{g^{-1}H_{1,i}g\cap H_{2,j}} \mathrm{res}^{g^{-1}H_{1,i}g}_{g^{-1}H_{1,i}g\cap H_{2,j}} g^{-1}\, \mathrm{pr}_i(z) \right).$$

We deduce the other formula in the same way. □

Remark A.3. We leave to the reader the generalisation to the case of groups $G_{1,i}$, $G_{2,j}$, and morphisms $\rho_{i,j} : G_{1,i} \to G_{2,j}$.

A.2. Decomposition "à la Künneth"

Recall, first, some useful lemmas

Lemma A.4. *Let $1 \to H \to G \to Q \to 1$, an exact sequence of groups and M a $\mathbb{Z}G$-module. We consider the usual action of Q on $\mathrm{H}_*(H;M)$. Let $Q' \triangleleft Q$. Suppose that the action of Q' on $\mathrm{H}_*(H;M)$ is trivial. Then, for all $n \in \mathbb{N}$, we have $\mathrm{H}_0(Q;\mathrm{H}_n(H;M)) \cong \mathrm{H}_0(Q/Q';\mathrm{H}_n(H;M))$.*

Lemma A.5. *Let $1 \to H \to G \xrightarrow{\pi} Q \to 1$ be an exact sequence of groups and M a $\mathbb{Z}G$-module. Let $K \le \mathrm{Z}(G)$ and set $Q' = \pi(K)$. Then Q' acts trivially on $\mathrm{H}_*(H;M)$.*

Proposition A.6. *Let G be a group, $A \le Z(G)$ and $H \triangleleft G$. Let*

$$\varphi : A \times H \to G$$

$$(a,h) \mapsto a.h,$$

and suppose that $\mathrm{Ker}\,(\varphi)$ *and* $\mathrm{Coker}\,(\varphi)$ *are torsion. Then, φ induces an isomorphism*

$$\mathrm{H}_n(G;\mathbb{Q}) \cong \bigoplus_{r+s=n} \mathrm{H}_0\left(\frac{G}{H};\mathrm{H}_r(H;\mathbb{Q})\right) \otimes \mathrm{H}_s(A;\mathbb{Q}) \quad \textit{for } n \ge 0.$$

Proof. As $\mathrm{Ker}\,(\varphi)$ and $\mathrm{Coker}\,(\varphi)$ are torsion, the LHS spectral sequences (with rational coefficients) associated to the following exact sequences,

$$1 \to \mathrm{Ker}\,(\varphi) \to A \times H \to \mathrm{Im}(\varphi) \to 1$$

$$1 \to \mathrm{Im}(\varphi) \to G \to \mathrm{Coker}\,(\varphi) \to 1$$

degenerate, and give the following isomorphisms,

$$\mathrm{H}_s(A \times H;\mathbb{Q}) \cong \mathrm{H}_s(\mathrm{Im}(\varphi);\mathbb{Q})$$

$$\mathrm{H}_0(\mathrm{Coker}\,(\varphi);\mathrm{H}_s(\mathrm{Im}(\varphi);\mathbb{Q})) \cong \mathrm{H}_s(G;\mathbb{Q})$$

for $s \geq 0$. Thanks to the following big diagram,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (A\cap H)\times H & \longrightarrow & A\times H & \xrightarrow{\pi\circ\varphi} & \frac{A}{A\cap H} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle j} & & \| \\
1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\pi} & \frac{G}{H} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| \\
 & & 1 & \longrightarrow & \mathrm{Coker}(\varphi) & \longrightarrow & \mathrm{Coker}(j) & \longrightarrow & 1
\end{array}$$

we get that $\mathrm{Coker}(\varphi) \cong \mathrm{Coker}(j)$, thus

$$\mathrm{H}_0(\mathrm{Coker}(\varphi); \mathrm{H}_s(\mathrm{Im}(\varphi); \mathbb{Q})) \cong \mathrm{H}_0(\mathrm{Coker}(j); \mathrm{H}_s(\mathrm{Im}(\varphi); \mathbb{Q})),$$

but as A is central in G, $A/A\cap H$ is central in G/H, hence, by (A.4) and (A.5), we have

$$\mathrm{H}_0(\mathrm{Coker}(j);\ \mathrm{H}_s(\mathrm{Im}(\varphi);\ \mathbb{Q})) \cong \mathrm{H}_0\left(\frac{G}{H};\ \mathrm{H}_s(\mathrm{Im}(\varphi); \mathbb{Q}\right).$$

But as $\mathrm{H}_s(A \times H; \mathbb{Q}) \cong \mathrm{H}_s(\mathrm{Im}(\varphi); \mathbb{Q})$, we see that

$$\mathrm{H}_0\left(\frac{G}{H}; \mathrm{H}_s(A \times H; \mathbb{Q})\right) \cong \mathrm{H}_s(G; \mathbb{Q}).$$

And since A is central in G, the action of G/H on the homology of A is trivial, and finally we get the isomorphism claimed. □

As a useful example, we have

Corollary A.7. *Let R be a commutative ring and p an integer with $p \geq 1$. Let*

$$\varphi_p : SL_p(R) \times R^{\times} \to GL_p(R)$$
$$(g, \lambda) \mapsto \lambda . g$$

Then φ_p induces an isomorphism

$$\mathrm{H}_n(GL_p(R); \mathbb{Q}) \cong \bigoplus_{r+s=n} \overline{\mathrm{H}}_r(SL_p(R); \mathbb{Q}) \otimes \mathrm{H}_s(R^{\times}; \mathbb{Q}) \text{ with } n \geq 0.$$

We also have, in the case of commutative H1-rings, the analogue of [12, (5.14) p. 122],

Proposition A.8. *For all n and p we have:*

(1) *The morphism* $\overline{\mathrm{H}}_n(SL_p(R);\mathbb{Q}) \to \mathrm{H}_n(GL_p(R);\mathbb{Q})$, *induced by* $SL_p(R) \hookrightarrow GL_p(R)$, *is into.*

(2) *If R is* H1 *then for all* $q \geq p$, *the injection* $SL_q(R) \rightarrowtail SL(R)$ *induces an isomorphism*

$$\overline{\mathrm{H}}_p(SL_q(R);\mathbb{Q}) \cong \mathrm{H}_p(SL(R);\mathbb{Q}).$$

(3) *If R is* H1, *then for all* $p \geq 1$, *we have*

$$\overline{\mathrm{H}}_p(SL_p(R);\mathbb{Q}) \cap \mathrm{Im}(\mathrm{H}_p(GL_{p-1}(R);\mathbb{Q}) \to \mathrm{H}_p(GL_p(R);\mathbb{Q}))$$
$$\cong \mathrm{Im}(\overline{\mathrm{H}}_p(SL_{p-1}(R);\mathbb{Q}) \to \overline{\mathrm{H}}_p(SL_p(R);\mathbb{Q})).$$

Remark A.9. We can apply the proof of Gerdes, word for word, by using the stability result of Guin [13]. Note that in (A.5(2)), the isomorphism is given through stability and by the fact that the action of $R^\times$ on the homology of $SL(R)$ is trivial. This last fact is a consequence of the existence of the morphism $GL(R) \to SL(R)$, $g \mapsto g \oplus \det(g^{-1})$.

A.2.1. Some effective topics in group homology

Let G be a group and A be an abelian subgroup of G. We have a morphism $\mathrm{cor}_A^G : \mathrm{H}_*(A;\mathbb{Z}) \to \mathrm{H}_*(G;\mathbb{Z})$ (cf. [7, Section III. 8–9, p. 78–80]). As A is abelian, we get a "shuffle" structure in homology who defined an injective map $\Psi : \bigwedge_{\mathbb{Z}}^* (A) \to \mathrm{H}_*(A;\mathbb{Z})$ (split if A is finitely generated) (cf. [7, Section 6.4(i), p. 123]). We write once again $\Psi : \bigwedge_{\mathbb{Z}}^n (A) \to \mathrm{H}_n(A;\mathbb{Z})$ and $\Xi = \mathrm{cor}_A^G \circ \Psi$. If $a_1,\dots,a_n$ are elements of A, we will denote $\mathbf{c}(a_1,\dots,a_n) = \Xi(a_1 \wedge \cdots \wedge a_n)$ the image of $a_1 \wedge \cdots \wedge a_n$, by Ξ, in $\mathrm{H}_n(G;\mathbb{Z})$. Then the properties of the exterior algebra give to us

Proposition A.10. *For all integer* $n \geq 1$, *if* $a_1,\dots,a_n,a_1'$ *are pairwise commuting elements of a group G. We have*

(1) $\mathbf{c}(a_1a_1',\dots,a_n) = \mathbf{c}(a_1,\dots,a_n) + \mathbf{c}(a_1',\dots,a_n)$.

(2) *If* $\sigma \in \mathfrak{S}_n$, $\mathbf{c}(a_{\sigma(1)},\dots,a_{\sigma(n)}) = \varepsilon(\sigma)\mathbf{c}(a_1,\dots,a_n)$, *where* $\varepsilon(\sigma)$ *denotes the signature of* σ.

(3) $\mathbf{c}()$ *is multilinear alternate.*

Proposition A.11. *Let R be a commutative* H1*-ring. We have a split short exact sequence*

$$0 \to \mathrm{K}_2(R) \xrightarrow{\mu} \mathrm{H}_2(GL_2(R);\mathbb{Z}) \xrightarrow{\overline{\det}} \bigwedge\nolimits_{\mathbb{Z}}^2 (R^\times) \to 0$$

with

$$\mu(\{a,b\}) = \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right),$$

split by

$$\mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) \mapsto \{c,b\} + \{d,a\},$$

and $\overline{\det}$ *split by*

$$a \wedge b \mapsto \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Proof. By [3, Theorem 1, p. 83] $\mathrm{H}_2(GL(R);\mathbb{Z}) \cong \mathrm{K}_2(R) \oplus \bigwedge^2_{\mathbb{Z}}(\mathrm{K}_1(R))$. As R is stable rank one, $\mathrm{K}_1(R) = R^\times$ and $\mathrm{K}_2(R) = \mathrm{H}_2(E(R);\mathbb{Z}) = \mathrm{H}_2(SL(R);\mathbb{Z})$. For all $p \geq 2$, we have an exact sequence

$$0 \to \overline{\mathrm{H}}_2(SL_p(R);\mathbb{Z}) \to \mathrm{H}_2(GL_p(R);\mathbb{Z}) \xrightarrow{\overline{\det}} \mathrm{H}_2(R^\times;\mathbb{Z}) \to 0. \tag{A.1}$$

Indeed, applying LHS to $1 \to SL_p(R) \to GL_p(R) \xrightarrow{\det} R^\times \to 1$, we get a spectral sequence

$$E^2_{s,t} = \mathrm{H}_s(R^\times; \mathrm{H}_t(SL_p(R);\mathbb{Z})) \Rightarrow \mathrm{H}_{s+t}(GL_p(R);\mathbb{Z}).$$

By ([9, (1.1.11) p. 9]), $\mathrm{H}_1(SL_p(R);\mathbb{Z}) = 0$ for all $p \geq 2$. Thus $E^2_{s,1} = 0 = E^\infty_{s,1}$ for all $s \geq 0$.

Moreover $E^\infty_{2,0} = E^2_{2,0} = \mathrm{H}_2(R^\times;\mathbb{Z})$. We deduce an exact sequence

$$0 \to E^\infty_{0,2} \to \mathrm{H}_2(GL_p(R);\mathbb{Z}) \to E^\infty_{2,0} \to 0,$$

because if $F_\bullet$ denotes the filtration of the abutment,

$$F_1\mathrm{H}_2(GL_p(R);\mathbb{Z}) = F_0\mathrm{H}_2(GL_p(R);\mathbb{Z}),$$

since $E^\infty_{1,1} = 0$ and $E^\infty_{0,2} = F_0\mathrm{H}_2(GL_p(R);\mathbb{Z})$. But $E^2_{2,1} = 0$ and thus no differential perturb $E^2_{0,2}$, hence $E^\infty_{0,2} = E^2_{0,2}$, then we get the exact sequence (A.1). As $R^\times$ acts trivially on the homology of $SL(R)$, at the infinite the sequence (A.1) becomes

$$0 \to \overline{\mathrm{H}}_2(SL(R);\mathbb{Z}) \to \mathrm{H}_2(GL(R);\mathbb{Z}) \xrightarrow{\overline{\det}} \mathrm{H}_2(R^\times;\mathbb{Z}) \to 0$$

and by combining the different isomorphisms, we get the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathrm{H}}_2(SL_2(R);\mathbb{Z}) & \longrightarrow & \mathrm{H}_2(GL_2(R);\mathbb{Z}) & \xrightarrow{\bar{d}} & \bigwedge^2_{\mathbb{Z}}(R^\times) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow j & & \| & & \\ 0 & \longrightarrow & \mathrm{K}_2(R) & \longrightarrow & \mathrm{H}_2(GL(R);\mathbb{Z}) & \xrightarrow{\bar{d}} & \bigwedge^2_{\mathbb{Z}}(R^\times) & \longrightarrow & 0 \end{array}$$

but by [13, Théorème 1, p. 33] j (induced by the stabilization morphism) is an isomorphism and by the "five lemma", $\mathrm{K}_2(R) \cong \widetilde{\mathrm{H}}_2(SL_2(R);\mathbb{Z})$. By the same arguments as in Barge [4, Lemme 3.2, p. 14] we note that the image of the symbol $\{x, y\}$ in $\mathrm{H}_2(GL_2(R);\mathbb{Z})$ is

$$\mathbf{c}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right)$$

(this morphism factorises through $\widetilde{\mathrm{H}}_2(SL_2(R);\mathbb{Z})$). The isomorphism from $\mathrm{K}_2(R) \oplus \bigwedge^2_{\mathbb{Z}}(R^\times)$ to $\mathrm{H}_2(GL_2(R);\mathbb{Z})$ is induced by

$$\{x, y\} + a \wedge b \mapsto \mathbf{c}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right). \qquad \square$$

Remark A.12. In (A.11), we can give explicitely the inverse on the cycles

$$\mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right).$$

As

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix},$$

we have

$$\begin{aligned} &\mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) \\ &\quad = \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right) \\ &\qquad + \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right) \end{aligned}$$

$$= \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$+ \mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$+ \mathbf{c}\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\right) + \mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right)$$

as the conjugation acts trivially in homology [7, (8.1), p. 79], conjugating by

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

gives

$$\mathbf{c}\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right) = \mathbf{c}\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and then

$$\mathbf{c}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right)$$

maps to $ab \wedge cd + (\{c,b\} + \{d,a\})$. Note that this is sufficient because the morphism $H_2(R^\times \times R^\times; \mathbb{Z}) \to H_2(GL_2(R); \mathbb{Z})$ is onto (see [18, 4.3.6, p. 206] for a proof in the case of infinite fields).

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